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Planar graphs are 9/2-colorable

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ABSTRACT

We show that every planar graph G has a 2-fold 9-coloring. In particular, this implies that G has fractional chromatic number at most $\frac{9}{2}$. This is the first proof (independent of the 4 Color Theorem) that there exists a constant $k < 5$ such that every planar G has fractional chromatic number at most k.

1. Introduction

All graphs in this paper are finite, loopless, and simple (parallel edges are forbidden). To fractionally color a graph G, we assign to each independent set in G a nonnegative weight, such that for each vertex v the sum of the weights on the independent sets containing v is 1. A graph G is *fractionally k-colorable* if G has such an assignment of weights where the sum of the weights is at most k. The minimum k such that G is fractionally k-colorable is its *fractional chromatic number*, denoted $\chi_f(G)$. (If we restrict the weight on each independent set to be either 0 or 1, then we return to the standard definition of chromatic number.) In 1997, Scheinerman and Ullman [13, p. 75] succinctly described the state of the art for fractionally coloring planar graphs. Not much has changed since then.

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The fractional analogue of the four-color theorem is the assertion that the maximum value of $\chi_f(G)$ over all planar graphs G is 4. That this maximum is no more than 4 follows from the four-color theorem itself, while the example of K_4 shows that it is no less than 4. Given that the proof of the four-color theorem is so difficult, one might ask whether it is possible to prove an interesting upper bound for this maximum without appeal to the four-color theorem. Certainly $\chi_f(G) \leq 5$ for any planar G, because $\chi(G) \leq 5$, a result whose proof is elementary. But what about a simple proof of, say, $\chi_f(G) \leq \frac{9}{2}$ for all planar G? The only result in this direction is in a 1973 paper of Hilton, Rado, and Scott [7] that predates the proof of the four-color theorem; they prove $\chi_f(G) < 5$ for any planar graph G, although they are not able to find any constant $c < 5$ with $\chi_f(G) < c$ for all planar graphs G. This may be the first appearance in print of the invariant χ_f.

In Section 2, we give exactly what Scheinerman and Ullman asked for—a simple proof that $\chi_f(G) \leq \frac{9}{2}$ for every planar graph G. In fact, this result is an immediate corollary of a stronger statement in our main theorem. Before we can express it precisely, we need another definition. A *k-fold ℓ-coloring* of a graph G assigns to each vertex a set of k colors, such that adjacent vertices receive disjoint sets, and the union of all sets has size at most ℓ. If G has a k-fold ℓ-coloring, then $\chi_f(G) \leq \frac{\ell}{k}$. To see this, consider the ℓ independent sets induced by the color classes; assign to each of these sets the weight $\frac{1}{k}$. Now we can state the theorem.

Main Theorem. *Every planar graph G has a 2-fold 9-coloring. In particular, $\chi_f(G) \leq \frac{9}{2}$.*

In an intuitive sense, the Main Theorem sits somewhere between the 4 Color Theorem and the 5 Color Theorem. It is certainly implied by the former, but it does not immediately imply the latter. The *Kneser graph $K_{n:k}$* has as its vertices the k-element subsets of $\{1, \ldots, n\}$ and two vertices are adjacent if their corresponding sets are disjoint. Saying that a graph G has a 2-fold 9-coloring is equivalent to saying that it has a homomorphism to the Kneser graph $K_{9:2}$. To claim that a coloring result for planar graphs is between the 4 and 5 Color Theorems, we would like to show that every planar graph G has a homomorphism to a graph H, such that H has clique number 4 and chromatic number 5. (Since K_4 can map into H, we know that H has clique number at least 4. And clique number less than 5 means our result is something more than just the 5 Color Theorem. The fact that H has chromatic number 5 means that our result implies the 5 Color Theorem.) Unfortunately, $K_{9:2}$ is not such a graph. It is easy to see that $\omega(K_{n:k}) = \lfloor n/k \rfloor$; so $\omega(K_{9:2}) = 4$, as desired. However, Lovász [9] showed that $\chi(K_{n:k}) = n - 2k + 2$; thus $\chi(K_{9:2}) = 9 - 2(2) + 2 = 7$. Fortunately, we can easily overcome this problem.

The *categorical product* (or *universal product*) of graphs G_1 and G_2, denoted $G_1 \times G_2$ is defined as follows. Let $V(G_1 \times G_2) = \{(u, v) | u \in V(G_1) \text{ and } v \in V(G_2)\}$; now (u_1, v_1) is adjacent to (u_2, v_2) if $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$. Let $H = K_5 \times K_{9:2}$. It is well-known [6] that if a graph G has a homomorphism to each of graphs G_1 and G_2, then G also has a homomorphism to $G_1 \times G_2$ (the image of each vertex in the product is just the products of its images in G_1 and G_2). The 5 Color Theorem says that every planar graph has a homomorphism to K_5; so if we prove that every planar graph G has a homomorphism to $K_{9:2}$, then we also get that G has a homomorphism to $K_5 \times K_{9:2}$.

It is easy to check that for any G_1 and G_2, we have $\omega(G_1 \times G_2) = \min(\omega(G_1), \omega(G_2))$ and $\chi(G_1 \times G_2) \leq \min(\chi(G_1), \chi(G_2))$. To prove this inequality, we simply color each vertex (u, v) of the product with the color of u in an optimal coloring of G_1, or the color of v in an optimal coloring of G_2. (It is an open problem whether this inequality always holds with equality [12].) When $H = K_5 \times K_{9:2}$ we get $\omega(H) = 4$ and $\chi(H) = 5$. Earlier work of Naserasr [10] and Nešetřil and Ossona de Mendez [11] also constructed graphs H, with $\omega(H) = 4$ and $\chi(H) = 5$, such that every planar graph G has a homomorphism to H; however, their examples had more vertices than ours. Naserasr gave a graph with size $63\binom{62}{4} = 35,144,235$ and the construction in [11] was still larger. In contrast, $|K_5 \times K_{9:2}| = 5\binom{9}{2} = 180$.

Wagner [14] characterized K_5-minor-free graphs. The Wagner graph is formed from an 8-cycle by adding an edge joining each pair of vertices that are distance 4 along the cycle. Wagner showed that every maximal K_5-minor-free graph can be formed recursively from planar graphs and copies of the Wagner graph by pasting along copies of K_2 and K_3 (see also [5, p. 175]). Since the Wagner graph is 3-colorable, it clearly has a 2-fold 9-coloring. To show that every K_5-minor-free graph is 2-fold 9-colorable, we color each smaller planar graph and copy of the Wagner graph, then permute colors so that the colorings agree on the vertices that are pasted together.

Hajós conjectured that every graph is $(k-1)$-colorable unless it contains a subdivision of K_k. This is known to be true for $k \leq 4$ and false for $k \geq 7$. The cases $k = 5$ and $k = 6$ remain unresolved. Since this problem seems difficult, we offer the following weaker conjecture.

Conjecture. *Every graph with no K_5-subdivision is 2-fold 9-colorable.*

An immediate consequence of the 4 Color Theorem is that every n-vertex planar graph has an independent set of size at least $\frac{n}{4}$, and this is best possible, as shown by the disjoint union of many copies of K_4. In 1968, Erdös [2] suggested that perhaps this corollary could be proved more easily than the full 4 Color Theorem. And in 1976, Albertson [1] showed, independently of the 4 Color Theorem, that every n-vertex planar graph has an independent set of size at least $\frac{2n}{9}$. (Recently [4], we strengthened this lower bound to $\frac{3n}{13}$.)

Albertson's proof inspired and heavily influenced our proof of the Main Theorem. The bulk of the work in our proof consists in showing that certain configurations are *reducible*, i.e., they cannot appear in a minimal counterexample to the theorem. The proof concludes via a discharging argument, where we show that every planar graph contains one of the forbidden configurations; hence, it is not a minimal counterexample.

Before the proof, we need a few definitions. A *k-vertex* is a vertex of degree k; similarly, a k^--vertex (resp. k^+-vertex) has degree at most (resp. at least) k. A k-neighbor of a vertex v is a k-vertex that is a neighbor of v; and k^--neighbors and k^+-neighbors are defined analogously. A k-cycle is a cycle of length k. A vertex set V_1 in a connected graph G is *separating* if $G \setminus V_1$ has at least two components. A cycle C is separating if $V(C)$ is separating. Finally, an *independent k-set* is an independent set (or stable set) of size k.

2. Fractional coloring of planar graphs

Now we prove our Main Theorem, that every planar graph has a 2-fold 9-coloring. Our proof uses the methods of reducibility and discharging. First, we prove that certain properties must hold for every minimal counterexample to the theorem (by "minimal" we mean having the fewest vertices and, subject to that, the fewest non-triangular faces). To conclude, we give a counting argument, via the discharging method, showing that every planar graph violates one of these properties. Thus, no minimal counterexample exists, so the theorem is true.

Hereafter, we write G to denote a minimal counterexample to the theorem. To remind the reader of this assumption, we will often refer to a *minimal* G. Whenever we say "a coloring", we mean a 2-fold 9-coloring. Note that G is a plane triangulation; otherwise, adding an edge contradicts our choice of G as having the fewest non-triangular faces.

Lemma 1. *A minimal G has no separating clique. Specifically, G has no separating 3-cycle.*

Proof. Suppose G has a separating clique X and let $C_1, \ldots, C_k$ be the components of $G \setminus X$. By minimality of $|G|$, we have colorings of $G[V(C_i) \cup X]$ for each $i \in \{1, \ldots, k\}$. Permute the colors on each subgraph $G[V(C_i) \cup X]$ so the colorings agree on X. Now identifying the copies of X in each $G[V(C_i) \cup X]$ gives a coloring of G, a contradiction. □

Although it was easy to prove, Lemma 1 will play a crucial role in our proof. We will often want to identify two neighbors u_1 and u_2 of a vertex v and color the smaller graph by minimality. To do so, we must ensure that u_1 and u_2 are indeed non-adjacent; these arguments typically use the fact that if u_1 and u_2 were adjacent, then u_1u_2v would be a separating 3-cycle.

Lemma 2. *A minimal G has minimum degree* 5.

Proof. Since G is a plane triangulation (and not C_3), it has minimum degree at least 3 and at most 5. If G contains a 3-vertex, then its neighbors induce a separating 3-cycle, contradicting Lemma 1. If G contains a 4-vertex v, then some pair of its neighbors are non-adjacent, since K_5 is non-planar. Form G' from G by deleting v and identifying a non-adjacent pair of its neighbors. Color G' by minimality, then lift the coloring back to G; only v is uncolored. Since two neighbors of v have the same colors, we can extend the coloring to G. □

The following fact will often allow us to extend a 2-fold 9-coloring to the uncolored vertices of an induced $K_{1,3}$. It will be useful in verifying that numerous configurations are forbidden from a minimal G. We will also often apply it when the uncolored subgraph is simply P_3.

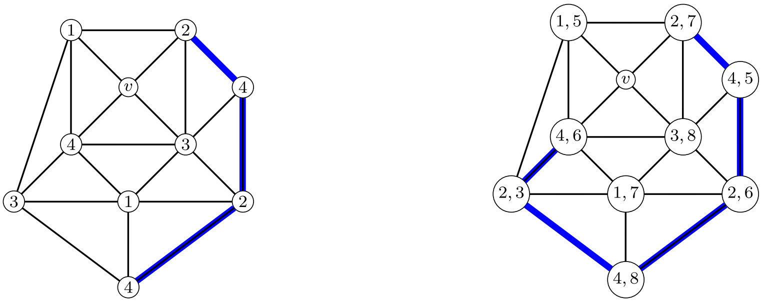
Fact 1. *Let $H = K_{1,3}$. If each leaf has a list of size 3 and the center vertex has a list of size 5, then we can choose 2 colors for each vertex from its lists such that adjacent vertices get disjoint sets of colors.*

Proof. Let v denote the center vertex and u_1, u_2, u_3 the leaves. Since $2|L(v)| > |L(u_1)| + |L(u_2)| + |L(u_3)|$, some color $c \in L(v)$ appears in $L(u_i)$ for at most one u_i. If such a u_i exists, then by symmetry, say it is u_1; now color v with c and some color not in $L(u_1)$. Otherwise color v with c and an arbitrary color. Now color each u_i arbitrarily from its at least 2 available colors. □

We use the same approach to prove each of Lemmas 3, 4, and 5. Our idea is to delete some vertices of G and identify others, to get a smaller planar graph G', which we color by minimality. In particular, when forming G' we identify some pairs of non-adjacent vertices of G that each have a common neighbor. When we lift the coloring of G' to G this means that some of the uncolored vertices will have neighbors with both colors the same, reducing the number of colors used on the neighborhood of each such uncolored vertex.

One early example of this technique is Kainen's proof [8] of the 5 Color Theorem. If G is a planar graph, then by Euler's Theorem, G has a 5^--vertex v. If $d(v) \leq 4$, then we 5-color $G - v$ by minimality; now, since $d(v) \leq 4$, we can extend the 5-coloring to v. Suppose instead that $d(v) = 5$. Since K_6 is non-planar, v has two neighbors u_1 and u_2 that are non-adjacent; form G' by deleting v and identifying u_1 and u_2, and again 5-color G' by minimality. To extend the 5-coloring to v, we note that even though $d(v) = 5$, at most four colors appear on the neighbors of v (since u_1 and u_2 have the same color). This completes the proof.

Because a minimal G has no separating 3-cycles, if vertices u_1 and u_2 have a common neighbor v and do not appear sequentially on the cycle induced by the neighborhood of v, then u_1 and u_2 are non-adjacent. The numeric labels in the figures denote pairs (or more) of vertices that are identified in G' when we delete any vertices labeled v, u_1, u_2 or u_3; vertices with the same numeric label get identified. Typically, it suffices to verify that the vertices receiving a common numeric label are pairwise non-adjacent. One potential complication is if two vertices that are drawn as distinct are in fact the same vertex. This usually cannot happen if the vertices have a common neighbor v, since then the degree of v would be too small. Similarly, it usually cannot happen if they are



(A) The 2, 4-path is blocked by the 1, 3-path. (B) The 2, 4-path gets through.

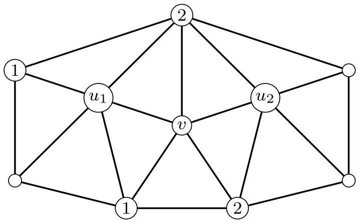
Fig. 1. The problem with Kempe chains for 2-fold coloring.

joined by a path of length three, since then we would get a separating 3-cycle. While this always gives a 3-cycle, it may be a facial 3-cycle and not a separating one. We say more about this possibility later.

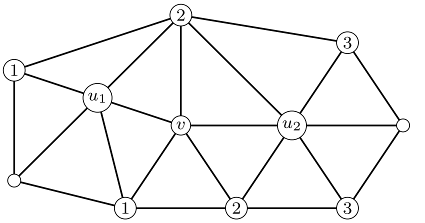
For 4-coloring, Birkhoff [3] showed how to exclude separating 4-cycles and 5-cycles. Excluding separating 4-cycles would simplify our arguments below since we would not need to worry about vertices at distance at most four being the same. The proof excluding 4-cycles for 4-coloring is quite easy, but it does not work in our context because standard Kempe chain arguments break down for 2-fold coloring. The problem is illustrated in Fig. 1. Fig. 1(A) shows the situation for 1-fold coloring; here the 13-path blocks the 24-path. Fig. 1(B) shows the situation for 2-fold coloring; here the 24-path can get through because on the 13-path a vertex has color 2 as well as color 3.

Lemma 3. *A minimal G has no 5-vertex with a 5-neighbor and a non-adjacent 6^--neighbor.*

Proof. We first consider the case where a 5-vertex v has non-adjacent 5-neighbors u_1 and u_2, as shown in Fig. 2(A). Recall that to form G', we delete v and all u_i and for each pair (or more) of vertices with the same label, we identify them. We typically know that G' is planar since it inherits an embedding from G. We must also verify that G' is loopless. For this we require that any two vertices with the same label are non-adjacent in G, and that any two vertices with distinct labels are distinct. This verification is generally routine, but tedious. So we include the details for the present lemma and mainly omit them hereafter. By assumption v, u_1, and u_2 are 5-vertices. Since v is drawn with precisely 5 neighbors, these neighbors must be distinct; similarly for neighbors of u_1 and of u_2. The vertices labeled 1 and 2 drawn at distance 3 must be distinct, since otherwise G has a separating 3-cycle, contradicting Lemma 1. For the same reason, the vertices labeled 1 must be non-adjacent, and also the vertices labeled 2 must be non-adjacent. Thus, G' is a loopless planar graph with fewer vertices than G. (Throughout the paper we suppress any parallel edges that appear while forming G'.)



(A) A 5-vertex, v, with non-adjacent 5-neighbors, u_1 and u_2.



(B) A 5-vertex, v, with a non-adjacent 5-neighbor, u_1, and 6-neighbor, u_2.

Fig. 2. The cases of Lemma 3.

We color G' by minimality, then lift the coloring to G. Now in G each u_i has a list of at least 3 colors and v has a list of at least 5 colors. So, by Fact 1, we can extend the coloring to G.

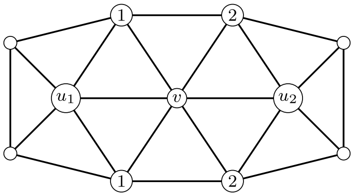
Now we consider the case where a 5-vertex v has a 5-neighbor, u_1, and a 6-neighbor, u_2, that are non-adjacent, as shown in Fig. 2(B). As in the previous case, all neighbors of v must be distinct, and similarly for neighbors of u_1 and of u_2. Each pair of vertices with a common label have a common neighbor for which they do not appear successively in its neighborhood. Thus, they cannot be adjacent, since this would create a separating 3-cycle. So every two vertices with the same label are non-adjacent. The left vertex labeled 1 must be distinct from the bottom vertices labeled 2 and 3, since otherwise G has a separating 3-cycle. The same is true for the bottom vertex labeled 1 and the top vertex labeled 3. The top vertices labeled 1 and 3 must be distinct, since otherwise the top vertex labeled 2 has degree 4, contradicting Lemma 2. Similarly, the bottom vertices labeled 1 and 3 must also be distinct. Thus, G' is a loopless planar graph with fewer vertices than G.

When we lift the coloring of G' to G, v has a list of size 5 and each of its uncolored neighbors has a list of size 3. Hence, by Fact 1, we can extend the coloring of G' to G. □

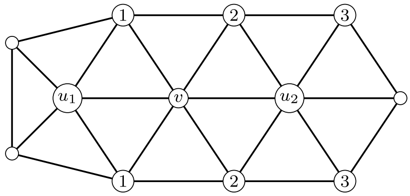
Lemma 4. *A minimal G has no 6-vertex with non-adjacent 6^--neighbors.*

Proof. Let v be a 6-vertex with two non-adjacent 6^--neighbors, u_1 and u_2. We have three possibilities for the degrees of these 6^--neighbors: two 5-vertices, a 5-vertex and a 6-vertex, and two 6-vertices. For each choice of degrees for the u_is, we have two possibilities for their relative location; they could be "across" from each other (at distance three along the cycle induced by the neighbors of v) or "offset" from each other (at distance two along the same cycle). This yields a total of six possibilities; the three across possibilities are shown in Fig. 3 and the three offset possibilities are shown in Fig. 4 (with two subcases for the third possibility).

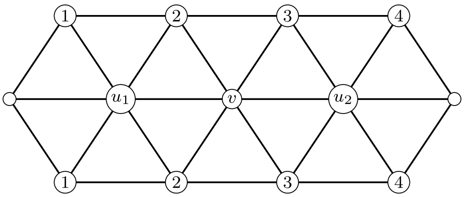
In Figs. 3(A, B), all of the vertices with numeric labels (those that will be identified in G') must be distinct, and each pair with the same label must be non-adjacent. The details are similar to the proof of Lemma 3. The only complication is in Fig. 3(C): a vertex labeled 1 might be the same as a vertex labeled 4 that is drawn at distance



(A) A 6-vertex, v, with non-adjacent 5-neighbors, u_1 and u_2, that are across from each other.

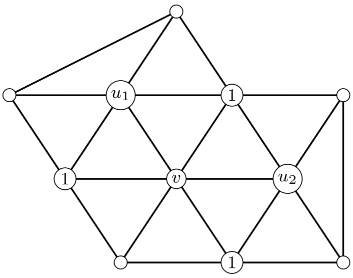


(B) A 6-vertex, v, with a non-adjacent 5-neighbor, u_1, and 6-neighbor, u_2, that are across from each other.

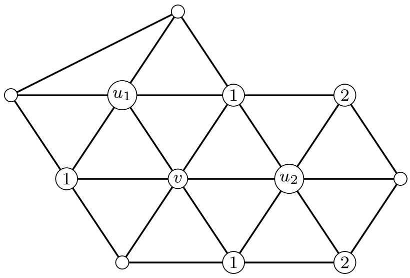


(C) A 6-vertex, v, with non-adjacent 6-neighbors, u_1 and u_2, that are across from each other.

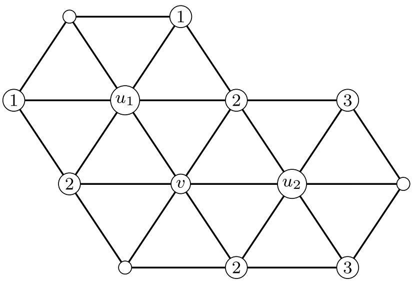
Fig. 3. The "across" cases of Lemma 4.



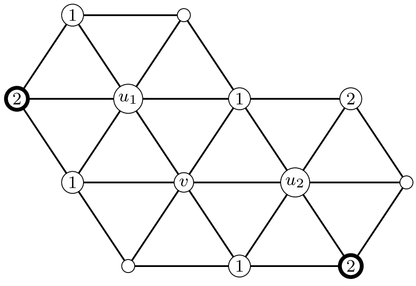
(A) A 6-vertex, v, with non-adjacent 5-neighbors, u_1 and u_2, that are offset from each other.



(B) A 6-vertex, v, with a non-adjacent 5-neighbor, u_1, and 6-neighbor, u_2, that are offset from each other.



(C) A 6-vertex, v, with non-adjacent 6-neighbors, u_1 and u_2, that are offset from each other (case i).



(D) A 6-vertex, v, with non-adjacent 6-neighbors, u_1 and u_2, that are offset from each other (case ii).

Fig. 4. The "offset" cases of Lemma 4.

four; call this vertex x. By symmetry, assume that x is formed by identifying the top vertex labeled 1 and the bottom vertex labeled 4. This is only a problem if also a vertex labeled 1 is adjacent to one labeled 4; so suppose this happens. Note that the top vertex labeled 4 cannot be adjacent to the bottom vertex labeled 1; they are on opposite sides of the cycle xu_1vu_2. So, again by symmetry, we assume that x is adjacent to the bottom vertex labeled 1. However, now we have a separating 3-cycle (consisting of x, its neighbor labeled 1, and their common neighbor u_1); this contradicts Lemma 1. This contradiction finishes the across cases.

Now we consider the three offset cases, shown in Fig. 4. As with the across cases, in Figs. 4(A, B) all vertices with numeric labels must be distinct, and vertices with the same label must be non-adjacent. (The details are similar to the proof of Lemma 3.) The only complication in is the third case, shown in Figs. 4(C, D): the vertices labeled 1 and 3 that are drawn at distance four in Fig. 4(C) might be the same; if so, then call this vertex x. In this case we switch to the identifications shown in Fig. 4(D); the two vertices drawn as bold are, in fact, the same vertex. Note that the two vertices labeled 1 that are drawn at distance three are non-adjacent, since they are separated by cycle u_1vu_2x. So we can verify that G' is loopless and planar, as in the proof of Lemma 3. This finishes the offset cases. □

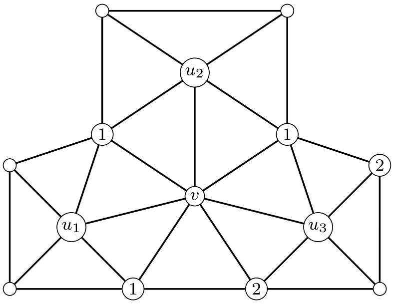
Lemma 5. *A minimal G has no 7-vertex with a 5-neighbor and two other 6^--neighbors such that all three are pairwise non-adjacent.*

Proof. Fig. 5(A) shows a 7-vertex with three pairwise non-adjacent 5-neighbors. We verify that G' is loopless as in the proof of Lemma 3.

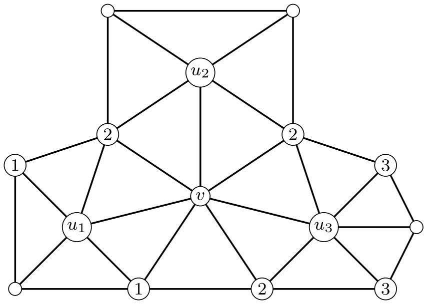
For Fig. 5(B), checking that G' is loopless is mostly straightforward, as in the proof of Lemma 3. The only possible problem is if one pair of vertices labeled 1 and 3 are actually the same vertex, while another pair labeled 1 and 3 are adjacent; these pairs must be disjoint, since otherwise we have a separating 3-cycle. (Note that the bottom vertices labeled 1 and 3 are distinct, since otherwise they have a common 3-neighbor. And they are also non-adjacent, since otherwise they either have a 4-neighbor or lie on a separating 3-cycle.) Hence, we need only consider the case where the vertices labeled 1 and 3 drawn at distance three are adjacent, and the other pair labeled 1 and 3 are the same vertex x. However, this is impossible, since then the adjacent pair are on opposite sides of the cycle u_1vu_3x.

For Fig. 5(C), we verify that G' is loopless as in the proof of Lemma 3.

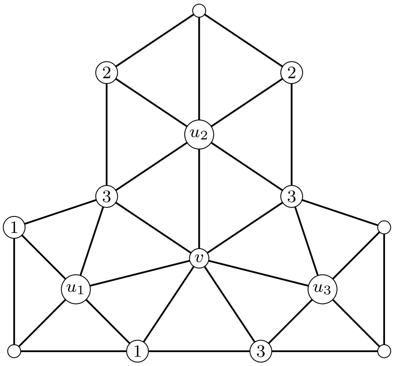
For Fig. 5(D), verifying that G' is loopless is mostly straightforward, as in the proof of Lemma 3. The only complication is that some vertex labeled 4 could be the same as one labeled 1 or 2. If a pair of vertices labeled 2 and 4 are the same, then it must be the pair that are drawn at distance 4; call this vertex x. In this case, we unlabel the vertices labeled 4 and label w_3 with 3. Now, thanks to x, it is easy to check that G' is loopless. So we may assume that vertices labeled 4 are not the same as those labeled 2.



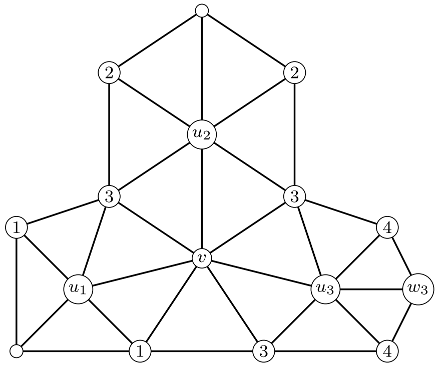
(A) A 7-vertex, v, with non-adjacent 5-neighbors, u_1, u_2, and u_3.



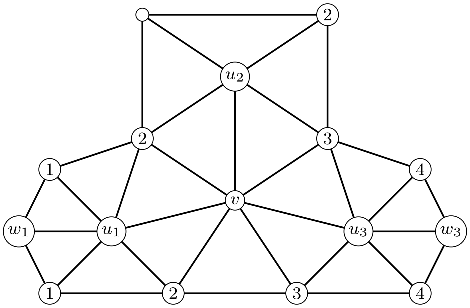
(B) A 7-vertex, v, with a 6-neighbor, u_3, and two 5-neighbors, u_1 and u_2, with all pairs of u_is non-adjacent.



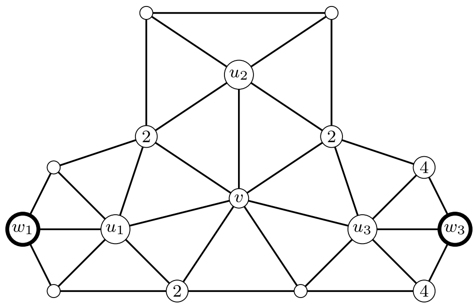
(C) A 7-vertex, v, with a 6-neighbor, u_2, and two 5-neighbors, u_1 and u_3, with all pairs of u_is non-adjacent.



(D) A 7-vertex, v, with a 5-neighbor, u_1, and two 6-neighbors, u_2 and u_3, with all pairs of u_is non-adjacent.



(E) A 7-vertex, v, with a 5-neighbor, u_2, and two 6-neighbors, u_1 and u_3, with all pairs of u_is non-adjacent (case i).



(F) A 7-vertex, v, with a 5-neighbor, u_2, and two 6-neighbors, u_1 and u_3, with all pairs of u_is non-adjacent (case ii).

Fig. 5. The cases of Lemma 5.

Suppose instead that a vertex labeled 4 is the same as one labeled 1; call this vertex y. This is only a problem if also some pair of vertices labeled 1 and 4 are adjacent. But this is impossible as follows. Since the pair of vertices labeled 1 have a common neighbor, they cannot be adjacent; similarly for the pair labeled 4. So the pairs that are identified and adjacent must be disjoint. Further, the identified pair must contain the left vertex labeled 1. If it is identified with the bottom vertex labeled 4, then the remaining vertices cannot be adjacent, since they are on opposite sides of the 4-cycle u_1vu_3y. If it is identified with the top vertex labeled 4, then the remaining pair cannot be adjacent since they have a common neighbor, and G would have a 3-vertex or a separating 3-cycle.

Now consider the final case, shown in Figs. 5(E, F). By horizontal symmetry and planarity, we assume the vertices labeled 2 that are drawn at distance 3 are neither the same nor adjacent, by reflecting across edge u_2v if necessary. (So the vertices labeled 1 and 2 drawn at distance 4 are distinct.) Hence, in forming G' we can identify all vertices labeled 2; we can also identify all vertices labeled 3. As in the proof of Lemma 3, it is straightforward to check that no vertex labeled 2 or 3 is the same as any other labeled vertex. So we only need to consider the vertices labeled 1 and 4. The only possible problem is if some pair of vertices labeled 1 and 4 that are drawn at distance four are actually the same vertex x. Further, this only causes difficulty if another pair labeled 1 and 4 are adjacent. So, suppose this is the case. Since G has no separating 3-cycle, it is easy to check that these pairs labeled 1 and 4 must be disjoint. This implies that w_1 is neither the same as, nor adjacent to, the top vertex labeled 2, since they are separated by a cycle through the pair labeled 1 and 4 that contains the top vertex labeled 1. If w_1 and w_3 are distinct, then we neglect the vertices labeled 1 and 4 altogether; instead we label w_1 as 2 and w_3 as 3. Due to the identified and adjacent pairs labeled 1 and 4, we can easily check that G' is loopless, as in the proof of Lemma 3. So assume that w_1 and w_3 are the same vertex, denoted by bold in Fig. 5(F). Now we switch the vertex identifications we use to form G'. Delete v, u_1, u_2, and u_3. Identify the two vertices labeled 4. Also identify the two neighbors of u_1 labeled 2, the top vertex that was labeled 3 (now 2), and $w_{1/3}$ (the bold vertex). Now it is straightforward to check that G' is loopless, as in the proof of Lemma 3. As usual, we color this smaller graph by minimality; when we lift this coloring to G, vertex v and each vertex u_i have enough available colors that we can extend the coloring by Fact 1. This finishes Figs. 5(E, F) and completes the proof of the lemma. □

Now we use discharging to prove that every planar graph has a 2-fold 9-coloring.

Main Theorem. *Every planar graph G has a 2-fold 9-coloring. In particular, $\chi_f(G) \le \frac{9}{2}$.*

Proof. The second statement follows from the first, which we prove now. Let G be a minimal counterexample to the theorem. We will use the discharging method with initial charge $d(v) - 6$ for each vertex v. We write $\mathrm{ch}(v)$ to denote the initial charge and $\mathrm{ch}^*(v)$ to denote the charge after redistributing. By Euler's Formula, $\sum_{v\in V(G)} \mathrm{ch}(v) = -12$. By assuming that G satisfies the conditions stipulated in Lemmas 1–5, we redistribute the charge (without changing its sum) so that every vertex finishes with nonnegative charge. This yields the obvious contradiction $-12 = \sum_{v\in V(G)} \mathrm{ch}(v) = \sum_{v\in V(G)} \mathrm{ch}^*(v) \ge 0$.

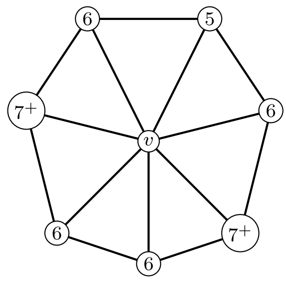


Fig. 6. A 7-vertex v gives no charge to any crowded 5-neighbor.

We need a few definitions. For a vertex v, let H_v denote the subgraph induced by the 5-neighbors and 6-neighbors of v. If some $w \in V(H_v)$ has $d_{H_v}(w) = 0$, then w is an *isolated* neighbor of v; otherwise w is a *non-isolated* neighbor. A non-isolated 5-neighbor of a vertex v is *crowded* (with respect to v) if it has two 6-neighbors in H_v. We use crowded 5-neighbors in the discharging proof to help ensure that 7-vertices finish with sufficient charge, specifically to handle the configuration in Fig. 6. We redistribute charge via the following four rules; they are applied simultaneously, wherever applicable.

(R1) Each 8^+-vertex gives charge $\frac{1}{2}$ to each isolated 5-neighbor and charge $\frac{1}{4}$ to each non-isolated 5-neighbor.
(R2) Each 7-vertex gives charge $\frac{1}{2}$ to each isolated 5-neighbor, charge 0 to each crowded 5-neighbor and charge $\frac{1}{4}$ to each remaining 5-neighbor.
(R3) Each 7^+-vertex gives charge $\frac{1}{4}$ to each 6-neighbor.
(R4) Each 6-vertex gives charge $\frac{1}{2}$ to each 5-neighbor.

To show that every vertex v finishes with nonnegative charge, we consider $d(v)$.

$\boldsymbol{d(v) \geq 8}$: We will show that v gives away charge at most $\frac{d(v)}{4}$. Since $d(v) \geq 8$, we have $\mathrm{ch}(v) = d(v) - 6 \geq \frac{d(v)}{4}$, so this will imply $\mathrm{ch}^*(v) \geq 0$. Rather than giving away charge by rules (R1) and (R3), instead let v give charge $\frac{1}{4}$ to each neighbor. Now let each isolated 5-neighbor w take also the charge $\frac{1}{4}$ that v gave to the neighbor that clockwise around v succeeds w. Now each neighbor of v has received at least as much charge as by rules (R1) and (R3) and v has given away charge $\frac{d(v)}{4}$. Thus, when v gives away charge according to rules (R1) and (R3), this charge is at most $\frac{d(v)}{4}$, so $\mathrm{ch}^*(v) \geq 0$.

$\boldsymbol{d(v) = 7}$: First, suppose that v has an isolated 5-neighbor w. Let $x, y \in N(v)$ be the two 7^+-vertices that are common neighbors of v and w. We will show that the total charge that v gives to $N(v) \setminus \{w, x, y\}$ is at most $\frac{1}{2}$. By Lemma 5, these four remaining vertices include at most two 6^--vertices. So, if v gives them a total of more than $\frac{1}{2}$, then one of them must be another isolated 5-neighbor. But now the final 6^--vertex must be distance at least 2 from each of the previous 5-neighbors, violating Lemma 5.

So instead assume that v has no isolated 5-neighbors. Thus, if v loses total charge more than 1, then it must have at least five 6^--neighbors that receive charge from it (since they each take charge $\frac{1}{4}$). So assume that $|H_v| \geq 5$. This implies that H_v consists of either (i) a 7-cycle or (ii) a single path or (iii) two paths. Recall from Lemma 4, that no 6-vertex has non-adjacent 6^--neighbors. This means that every vertex of degree 2 in H_v is a 5-vertex; in other words, every vertex on a cycle or in the interior of a path in H_v is a 5-vertex.

Now in each of cases (i)–(iii), H_v has an independent 3-set containing at least one 5-vertex, which violates Lemma 5; the only exception is if H_v consists of a path on two vertices and a path on three vertices, and the only 5-vertex is the internal vertex on the longer path. However, in this case the 5-vertex is a crowded neighbor of v, as in Fig. 6, so it receives no charge from v. Thus, $\text{ch}^*(v) \geq 0$.

$\boldsymbol{d(v) = 6}$: By Lemma 4, we know that v has at most two 6^--neighbors (and if exactly two, then they are adjacent). Now (R3) implies that $\text{ch}^*(v) \geq 0 + 4(\frac{1}{4}) - 2(\frac{1}{2}) = 0$.

$\boldsymbol{d(v) = 5}$: If v has at least two 6-neighbors, then $\text{ch}^*(v) \geq -1 + 2(\frac{1}{2}) = 0$; so assume that v has at most one 6-neighbor. Now if v has at least four 6^+-neighbors, then $\text{ch}^*(v) \geq -1+4(\frac{1}{4}) = 0$ (since v has at most one 6-neighbor, v is not a crowded neighbor for any of its 7-neighbors); so v must have at least two 5-neighbors. By Lemma 3, these 5-neighbors must be adjacent and v has no 6-neighbors. But now one of v's three 7^+-neighbors sees v as an isolated 5-neighbor, so sends v charge $\frac{1}{2}$. Thus, $\text{ch}^*(v) \geq -1 + \frac{1}{2} + 2(\frac{1}{4}) = 0$. This completes the proof. □

A natural question is whether our theorem could be strengthened to show that every planar graph has a t-fold s-coloring, for some pair (s,t) with $\frac{s}{t} < \frac{9}{2}$. Such results are true for every pair (s,t) with $\frac{s}{t} \geq 4$, since they follow from the 4 Color Theorem (because the Kneser graph $K_{s:t}$ contains K_4). But any proof of such a result must differ significantly from the proof of the Main Theorem. In particular, none of our reducibility proofs, with the exceptions of those for separating triangles and 4^--vertices, remain valid for any pair (s,t) with $\frac{s}{t} < \frac{9}{2}$. Recall that the proofs of Lemmas 3–5 all crucially relied on Fact 1. We show that to prove an analogue of this fact, even for $K_{1,2}$ (rather than $K_{1,3}$) requires $\frac{s}{t} \geq \frac{9}{2}$.

Consider an analogue of Lemma 3, 4, or 5 for t-fold s-coloring. First we contract, color the smaller graph by minimality, and lift the coloring to G. Now the center vertex, v, has list size $s-2t$, and each leaf, u_i, has list size $s-3t$. Let $a = s-2t$ and $b = s-3t$. Consider the list assignment $L(u_1) = \{1,\ldots,b\}$, $L(u_2) = \{a-b+1,\ldots,a\}$, and $L(v) = \{1,\ldots,a\}$. Every t-fold coloring from these lists uses at most $b-(a-b) = 2b-a$ common colors on u_1 and u_2, so uses at least $2t-(2b-a)$ distinct colors on u_1 and u_2 (note that $2b-a = 2(s-3t)-(s-2t) = s-4t \geq 0$, since we must be able to s-fold t-color K_4). So, to color v, we must have $a-(2t-(2b-a)) \geq t$, which means $b \geq \frac{3}{2}t$. Thus, $s-3t = b \geq \frac{3}{2}t$, so $\frac{s}{t} \geq \frac{9}{2}$.

Acknowledgments

As we mentioned in the introduction, the ideas in this paper come largely from Albertson's proof [1] that planar graphs have independence ratio at least $\frac{2}{9}$. In fact, many of the reducible configurations that we use here are special cases of the reducible configurations in that proof. We very much like that paper, and so it was a pleasure to be able to extend Albertson's work. We also thank a referee for reading the manuscript very carefully and catching numerous inaccuracies.

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