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A NOTE ON REED'S CONJECTURE*

LANDON RABERN†

Abstract. In [*J. Graph Theory*, 27 (1998), pp. 177–212], Reed conjectures that every graph satisfies $\chi \leq \lceil \frac{\omega+\Delta+1}{2} \rceil$. We prove that this holds for graphs with disconnected complement. Combining this fact with a result of Molloy proves the conjecture for graphs satisfying $\chi > \lceil \frac{n}{2} \rceil$. Generalizing this we prove that the conjecture holds for graphs satisfying $\chi > \frac{n+3-\alpha}{2}$. It follows that the conjecture holds for graphs satisfying $\Delta \geq n + 2 - (\alpha + \sqrt{n+5-\alpha})$. In the final section, we show that if G is an even order counterexample to Reed's conjecture, then $\overline{G}$ has a 1-factor.

Key words. graph coloring, Reed's conjecture, chromatic number

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1. Introduction.

Reed's conjecture states that every graph satisfies

$$\chi \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil. \tag{1}$$

The weaker statement that every graph satisfies

$$\chi \leq \frac{\omega + n}{2} \tag{2}$$

follows easily by observing that in any optimal coloring the set of vertices in singleton color classes induces a clique.

We prove the following intermediate result.

Key lemma. $\chi \leq \frac{\omega + \frac{n+\Delta+1}{2}}{2}$.

Using this result it is easy to prove Reed's conjecture for graphs with disconnected complement. Combining this fact with a result of Molloy tells us that Reed's conjecture holds for graphs satisfying $\chi > \lceil \frac{n}{2} \rceil$. Building on this result, we show that the conjecture holds for graphs satisfying $\chi > \frac{n+3-\alpha}{2}$. With a bit more work we determine that the conjecture holds for graphs satisfying $\Delta \geq n + 2 - \left(\alpha + \sqrt{n+5-\alpha}\right)$.

Further analysis allows us to prove some related results.

In all that follows, *graph* will mean a finite simple graph with nonempty vertex set. Let $\mathbb{G}$ be the collection of all graphs. Let $R_t \subseteq \mathbb{G}$ be the graphs satisfying $\chi \leq \frac{1}{2}(\omega + \Delta + 1) + t$.

Definition 1. *Given graphs A and B, their* join $A + B$ *is the graph with vertex set $V(A) \cup V(B)$ and edge set $E(A) \cup E(B) \cup \{ab \mid a \in V(A), b \in V(B)\}$. Also, if X and Y are collections of graphs, we let $X + Y = \{A + B \mid A \in X, B \in Y\}$.*

First, we present a few basic facts about joins.

Lemma 1.1. *Let A and B be graphs. Then*
(a) $|A + B| = |A| + |B|$,
(b) $\omega(A + B) = \omega(A) + \omega(B)$,
(c) $\chi(A + B) = \chi(A) + \chi(B)$,
(d) $\Delta(A + B) = \max\{\Delta(A) + |B|, |A| + \Delta(B)\}$.

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†3200 Carbon Place, S-208, Boulder, CO 80301 (landon.rabern@gmail.com).

Proof. These all follow immediately from the definitions. ◻

2. Reed's conjecture for joins of graphs.

PROPOSITION 2.1. *Let A and B be graphs. Then $A + B \in R_0$.*

Proof. Applying the Key Lemma to A and B and adding the inequalities yields

$$\chi(A) + \chi(B) \le \frac{1}{2}\left(\omega(A) + \omega(B) + \frac{\Delta(A) + |B| + |A| + \Delta(B) + 2}{2}\right).$$

Using Lemma 1.1(b), (c), and (d), this becomes

$$\chi(A + B) \le \frac{1}{2}\left(\omega(A + B) + \frac{2\Delta(A + B) + 2}{2}\right) = \frac{1}{2}(\omega(A + B) + \Delta(A + B) + 1).$$

Hence $A + B \in R_0$. ◻

LEMMA 2.2. $\mathbb{G} + R_t \subseteq R_t$ *for all* $t \in \mathbb{R}$.

Proof. Fix $t \in \mathbb{R}$. Let $G \in \mathbb{G}$ and $H \in R_t$. Applying (2) to G gives

$$\chi(G) \le \tfrac{1}{2}(\omega(G) + |G|).$$

Also, since $H \in R_t$,

$$\chi(H) \le \tfrac{1}{2}(\omega(H) + \Delta(H) + 1) + t.$$

Adding these inequalities and applying Lemma 2(b) and (c) gives

$$\chi(G + H) \le \tfrac{1}{2}(\omega(G + H) + |G| + \Delta(H) + 1) + t.$$

Now Lemma 1.1(d) gives $|G| + \Delta(H) \le \Delta(G + H)$ and the result follows. ◻

Remark. We will use Lemma 2.2 in our proof of the Key Lemma; we note that we did not use the Key Lemma in its proof. It should also be noted that Lemma 2.2 just says that R_t is an ideal in the Abelian semigroup $(\mathbb{G}, +)$.

3. Proof of the Key Lemma. Combining Lemma 2.2 with the following two results of Molloy (see [2]) allows us to prove Reed's conjecture for graphs satisfying $\alpha = 2$. We note that these lemmas were proved by Gallai decades before Molloy rediscovered them (see [1]).

LEMMA 3.1. *Let G be a graph with $\chi(G) > \lceil \frac{|G|}{2} \rceil$. Then there exists $X \subseteq V(G)$ such that $\overline{G - X}$ is disconnected and $\chi(G - X) = \chi(G)$.*

As a corollary we get the following.

LEMMA 3.2. *If G is a vertex critical graph with $\chi(G) > \lceil \frac{|G|}{2} \rceil$, then $\overline{G}$ is disconnected.*

PROPOSITION 3.3. *If G is a graph with $\alpha(G) \le 2$, then $G \in R_{\frac{1}{2}}$.*

Proof. Assume this is not the case and let G be a counterexample with the minimum number of vertices, say $|G| = n$. Since $\alpha(G) \le 2$, we see that $V(G) \setminus N(v) \cup \{v\}$ induces a clique for each $v \in V(G)$. Hence $\omega(G) \ge n - \delta(G) - 1$ which gives

$$\Delta(G) + \omega(G) + 1 \ge n.$$

So, since $G \notin R_{\frac{1}{2}}$, we have $\chi(G) > \lceil \frac{n}{2} \rceil$.

Now, using minimality of G, we see that G is vertex critical. Thus $\overline{G}$ is disconnected by Lemma 3.2. Hence we have $m \ge 2$ and (nonempty) graphs $C_1, \dots, C_m$ such that $G = C_1 + \cdots + C_m$. But, for $1 \le i \le m$, minimality of G gives $C_i \in R_{\frac{1}{2}}$ since $\alpha(C_i) \le \alpha(G) \le 2$ and $|C_i| < n$. Hence $G = C_1 + \cdots + C_m \in R_{\frac{1}{2}}$ by Lemma 2.2. This contradiction completes the proof. $\square$

To prove our Key Lemma, we will also need the following result from [3].

THEOREM 3.4. *Let $I_1, \ldots, I_m$ be disjoint independent sets in a graph G. Then*

$$\chi(G) \le \frac{1}{2}\left(\omega(G) + |G| - \sum_{j=1}^{m} |I_j| + 2m - 1\right). \tag{3}$$

Remark. If we increase the bound in this theorem by $\frac{1}{2}$ we get a corollary of (2).

To apply Theorem 3.4 and Proposition 3.3 we need a definition.

DEFINITION 2. *Let G be a graph and r a positive integer. A collection of disjoint independent sets in G, each with at least r vertices, will be called an r*-greedy *partial coloring of G. A vertex of G is said to be* missed *by a partial coloring just in case it appears in none of the independent sets.*

We use the following consequence of Theorem 3.4.

COROLLARY 3.5. *Let G be a graph which is not complete and let C be an r-greedy partial coloring of G. Then*

$$\chi(G) \le \tfrac{1}{2}(\omega(G) + |G| - (r-2)|C| - 1).$$

Combining Corollary 3.5 for $r = 3$ with the following lemma yields the Key Lemma.

LEMMA 3.6. *Let G be a graph, and of all* 3*-greedy partial colorings of G, let C be one that misses the minimum number of vertices. Then*

$$\chi(G) \le \tfrac{1}{2}(\omega(G) + \Delta(G) + 1) + \tfrac{|C|+1}{2}.$$

Proof. The first case to consider is when C misses zero vertices. In this case, C is a proper coloring of G, and hence $\chi(G) \le |C|$. Thus

$$\chi(G) \le \tfrac{1}{2}(\chi(G) + |C|) \le \tfrac{1}{2}(\Delta(G) + 1 + |C|) \le \tfrac{1}{2}(\omega(G) + \Delta(G) + 1) + \tfrac{|C|+1}{2}.$$

Otherwise, C misses at least one vertex, and by the minimality condition placed on C, each vertex missed by C must be adjacent to at least one vertex in each element of C. Hence $\Delta(G - \cup C) \le \Delta(G) - |C|$. In addition, $\alpha(G - \cup C) \le 2$. Thus applying Proposition 3.3 to $G - \cup C$ yields

$$\begin{aligned}
\chi(G) &\le |C| + \chi(G - \cup C) \\
&\le |C| + \tfrac{1}{2}(\omega(G - \cup C) + \Delta(G - \cup C) + 1) + \tfrac{1}{2} \\
&\le |C| + \tfrac{1}{2}(\omega(G) + \Delta(G) - |C| + 1) + \tfrac{1}{2} \\
&= \tfrac{1}{2}(\omega(G) + \Delta(G) + 1) + \tfrac{|C|+1}{2}. \qquad \square
\end{aligned}$$

4. Further details. Proposition 2.1 allows us to prove Reed's conjecture for graphs satisfying $\chi > \lceil \frac{n}{2} \rceil$.

COROLLARY 4.1. *If G is a graph with $\chi(G) > \lceil \frac{|G|}{2} \rceil$, then $G \in R_0$.*

Proof. Let G be a graph with $\chi(G) > \lceil \frac{|G|}{2} \rceil$. Then, by Lemma 3.1, we have $X \subseteq V(G)$ such that $\overline{G - X}$ is disconnected and $\chi(G - X) = \chi(G)$. Since $\overline{G - X}$

is disconnected, there exist graphs A and B such that $G - X = A + B$. Hence, by Proposition 2.1,

$$\chi(G) = \chi(G - X) \leq \tfrac{1}{2}(\omega(G - X) + \Delta(G - X) + 1) \leq \tfrac{1}{2}(\omega(G) + \Delta(G) + 1),$$

whence $G \in R_0$. ∎

We can generalize the above corollary by passing from R_0 to $R_{\frac{1}{2}}$.

Corollary 4.2. *If G is a graph with $\chi(G) > \frac{|G|+3-\alpha(G)}{2}$, then $G \in R_{\frac{1}{2}}$.*

Proof. Let G be a graph with $\chi(G) > \frac{|G|+3-\alpha(G)}{2}$ and I an independent set in G with $\alpha(G)$ vertices. Put $H = G \setminus I$. Then

$$\begin{aligned} \chi(H) &\geq \chi(G) - 1 \\ &> \frac{|G| + 3 - \alpha(G)}{2} - 1 \\ &= \frac{|G| + 1 - \alpha(G)}{2} \\ &= \frac{|H| + 1}{2}. \end{aligned}$$

Hence, by Corollary 4.1, we have

$$\chi(H) \leq \frac{\omega(H) + \Delta(H) + 1}{2}.$$

But I is a maximal independent set, and hence each vertex of H is adjacent to at least one vertex in I. In particular, $\Delta(H) \leq \Delta(G) - 1$, whence

$$\chi(G) \leq \chi(H) + 1 \leq \frac{\omega(H) + \Delta(G) - 1 + 1}{2} + 1 \leq \frac{\omega(G) + \Delta(G) + 1}{2} + \frac{1}{2}.$$

The theorem follows. ∎

Remark. After noting that the conditions on the graph order in both Corollaries 4.1 and 4.2 are sufficient to force the existence of a doubly critical edge in a minimum counterexample to Reed's conjecture, it is natural to ask if similar results can be proved under the weaker condition that the graph contains a doubly critical edge. This is indeed the case, as is shown in [4] and [5]. The former paper generalizes the method of proof using singleton color classes that was sketched in the introduction as justification of (2). It is proved that a graph with more than $\frac{\omega}{2}$ singleton color classes satisfies Reed's conjecture. The latter paper proves Reed's conjecture for claw-free graphs containing a doubly critical edge and shows that general graphs containing a doubly critical edge satisfy $\chi \leq \frac{1}{3}\omega + \frac{2}{3}(\Delta + 1)$.

Corollary 4.3. *Let G be a graph and $t \geq \frac{1}{2}$. If $G \notin R_t$, then $\Delta(G) + 1 \leq |G| - 2t - \omega(G) - \alpha(G) + 2$.*

Proof. This is an immediate consequence of the previous corollary. ∎

Lemma 4.4. *Let G be a graph with $\alpha(G) \leq 2$. Then $\omega(G)^2 + \omega(G) \geq |G|$.*

Proof. Let K be a maximal clique in G. Then each vertex of $G - K$ is nonadjacent to at least one vertex in K, and hence some vertex $v \in K$ is nonadjacent to at least $\frac{|G-K|}{|K|}$ vertices. Since $\alpha(G) \leq 2$, the vertices nonadjacent to v form a clique, whence $\omega(G) \geq \frac{|G-K|}{|K|} = \frac{|G|-\omega(G)}{\omega(G)}$, which yields

$$\omega(G)^2 + \omega(G) \geq |G|. \quad ∎$$

Proposition 4.5. *If G is a graph with $\Delta(G) \geq |G|+2-\left(\alpha(G)+\sqrt{|G|+5-\alpha(G)}\right)$, then $G \in R_{\frac{1}{2}}$.*

Proof. We prove the contrapositive. Let G be a graph with n vertices, maximal degree Δ, clique number ω, and independence number α such that $G \notin R_{\frac{1}{2}}$. Let I be a maximal independent set in G. Let S be a maximal collection of disjoint 3-vertex independent sets of $G - I$. Since $\alpha(G - (\cup S) \cup I) \leq 2$, we may apply Lemma 4.4 to get $\omega^2 + \omega \geq |G - (\cup S) \cup I| = n - \alpha - 3|S|$. Hence

$$|S| \geq \frac{n - \alpha - (\omega^2 + \omega)}{3}. \tag{4}$$

Now, combining the fact that $G \notin R_{\frac{1}{2}}$ with Corollary 3.5, we have $n - \alpha - |S| + 1 > \Delta + 2$. Putting this together with (4), we have

$$n - \alpha - \Delta - 1 > |S| \geq \frac{n - \alpha - (\omega^2 + \omega)}{3},$$

which implies that

$$\Delta < \frac{2n + \omega^2 + \omega - 2\alpha - 3}{3}. \tag{5}$$

By Corollary 4.3, $\omega \leq n - \Delta - \alpha$. Plugging this into (5) and doing a little algebra, we find that $\Delta < n - \alpha + 2 - \sqrt{n - \alpha + 5}$. This completes the proof of the contrapositive. ∎

Corollary 4.6. *If G is a graph with $\Delta(G) \geq |G| - \left(1 + \sqrt{|G| + 2}\right)$, then $G \in R_{\frac{1}{2}}$.*

Proof. Let G be a graph with $\Delta(G) \geq |G| - (1 + \sqrt{|G| + 2})$. If $\alpha(G) \leq 2$, then $G \in R_{\frac{1}{2}}$ by Proposition 3.3. Otherwise, $\alpha(G) \geq 3$ and $G \in R_{\frac{1}{2}}$ by Proposition 4.5. ∎

5. Graphs without perfect matchings in their complements. The bound on Δ in Corollary 4.3 works only for $t \geq \frac{1}{2}$. In this section we need bounds on Δ that work for $t = 0$ as well. Using a single independent set of maximal order in Theorem 3.4, we deduce the following.

Corollary 5.1. *Let G be a graph. Then*

$$\chi(G) \leq \tfrac{1}{2}(\omega(G) + |G| - \alpha(G) + 1).$$

Corollary 5.2. *Let G be a graph and $t \in \frac{1}{2}\mathbb{Z}$. If $G \notin R_t$, then $\Delta(G) + 1 \leq |G| - 2t - \alpha(G)$.*

Proof. Assume $G \notin R_t$. Applying Corollary 5.1 gives

$$\tfrac{1}{2}(\omega(G) + \Delta(G) + 1) + t < \chi(G) \leq \tfrac{1}{2}(\omega(G) + |G| - \alpha(G) + 1).$$

The corollary follows. ∎

Corollary 5.3. *Let G be a graph and $t \geq 0$. If $G \notin R_t$, then $\Delta(G) + 1 \leq |G| - 2t - \omega(G)$.*

Proof. This is an immediate consequence of Corollary 4.1. ∎

Note that for $t \geq \frac{1}{2}$ in Corollary 5.2, we must have $\alpha(G) \geq 3$ by Proposition 3.3, so the lemma gives $\Delta(G) + 1 \leq |G| - 2t - 3$.

Lemma 5.4. *If $k \geq 2$ and $G_1, \dots, G_k$ are graphs with $\Delta(G_i) + 1 \leq |G_i| - 3$ for each i, then*

$$G_1 + \cdots + G_k \in R_{2-k}.$$

Proof. Assume this is not the case and let $G_1, \dots, G_k$ constitute a counterexample with the smallest k. Then, by Proposition 2.1, $k > 2$. Set $D = G_1 + \cdots + G_{k-1}$. Note that $D \in R_{2-(k-1)} = R_{3-k}$ by the minimality of k. Let $t \in \frac{1}{2}\mathbb{Z}$ be minimal such that $G_k \in R_t$. Since $G_k \notin R_{t-\frac{1}{2}}$, using Corollary 5.2 for $t \geq 1$ and the fact that $\Delta(G_k) + 1 \leq |G_k| - 3$ for $t \leq \frac{1}{2}$, we find that $\Delta(G_k) + 1 \leq |G_k| - 2t - 2$. We have

$$\begin{aligned}
\chi(D + G_k) &= \chi(D) + \chi(G_k) \\
&\leq \tfrac{1}{2}(\omega(D) + \omega(G_k) + \Delta(D) + \Delta(G_k) + 2) + 3 - k + t \\
&= \tfrac{1}{2}(\omega(D + G_k) + \Delta(D) + \Delta(G_k) + 2) + 3 - k + t \\
&= \tfrac{1}{2}(\omega(D + G_k) + \Delta(D) + |G_k| + 1 + \Delta(G_k) - |G_k| + 1) + 3 - k + t \\
&\leq \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + \tfrac{1}{2}(\Delta(G_k) - |G_k| + 1) + 3 - k + t \\
&\leq \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + \tfrac{1}{2}(-2t - 3 + 1) + 3 - k + t \\
&= \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + 2 - k.
\end{aligned}$$

Hence $G_1 + \cdots + G_k \in R_{2-k}$, contradicting our assumption. ∎

The hypotheses of this lemma can be weakened, but we do not use the following stronger lemma in what follows.

LEMMA 5.5. *If $k \geq 2$ and $G_1, \dots, G_k$ are graphs, which are not 5-cycles, with $\Delta(G_i) + 1 \leq |G_i| - 2$ for each i, then*

$$G_1 + \cdots + G_k \in R_{2-k}.$$

Proof. This is similar to the proof of Lemma 5.4. Graphs with $\Delta(G_i)+1 \leq |G_i|-3$ matter only if $G_i \in R_{\frac{1}{2}} \setminus R_0$. Corollary 5.3 shows that such graphs have $\omega(G_i) \leq 2$ and Corollary 5.2 shows they have $\alpha(G) \leq 2$. Thus they have order less than 6 and we see that the only one that breaks the lemma is C_5. ∎

DEFINITION 3. *The* matching number *of a graph G, denoted $\nu(G)$, is the number of edges in a maximum matching of G.*

PROPOSITION 5.6. *Let G be a graph. If $\nu(\overline{G}) < \lfloor \frac{|G|}{2} \rfloor$, then $G \in R_0$.*

Proof. Assume $\nu(\overline{G}) < \lfloor \frac{|G|}{2} \rfloor$. Then, by Tutte's theorem, we have $X \subseteq V(G)$ such that $\overline{G - X}$ has at least m odd components, where $m \geq |X| + 2$ if $|G|$ is even and $m \geq |X| + 3$ if $|G|$ is odd. Hence we have graphs $G_1, \dots, G_m$ such that $G - X = G_1 + \cdots + G_m$. Note that by picking one vertex from each component we induce a clique. Hence $\omega(G) \geq m$. To get a contradiction, assume $G \notin R_0$. First, assume there is some G_i for which $\Delta(G_i) + 1 \geq |G_i| - 2$; then

$$\begin{aligned}
\Delta(G) + 1 \geq \Delta(G - X) + 1 &\geq |G_1| + \cdots + |G_{i-1}| + \Delta(G_i) + |G_{i+1}| + \cdots + |G_m| + 1 \\
&\geq |G| - |X| - 2.
\end{aligned}$$

Since $G \notin R_0$,

$$\chi(G) > \tfrac{1}{2}(\omega(G) + \Delta(G) + 1) \geq \tfrac{1}{2}(m + (|G| - |X| - 2)) \geq \left\lceil \tfrac{|G|}{2} \right\rceil.$$

Hence $G \in R_0$ by Corollary 4.1! Thus we may assume $\Delta(G_i)+1 \leq |G_i|-3$ for each i. Now Lemma 5.4 yields $G-X \in R_{-|X|}$, whence $G \in R_0$. This contradiction completes the proof. ∎

COROLLARY 5.7. *Let G be an even order graph. If $G \notin R_0$, then $\overline{G}$ has a 1-factor.*

DEFINITION 4. *A graph is called* matching covered *if every edge participates in a perfect matching.*

COROLLARY 5.8. *Let G be an even order graph with $G \notin R_1$. Then $\overline{G}$ is matching covered.*

Lemma 5.4 can be generalized.

LEMMA 5.9. *Let $m \in \mathbb{N}$. Let $k \geq 2$ and $G_1, \dots, G_k$ be graphs such that $|G_i| < r(m,m) \Rightarrow G_i \in R_{\frac{1}{2}}$. If $\Delta(G_i) + 1 \leq |G_i| - m$ for each i, then*

$$G_1 + \cdots + G_k \in R_{(m-1)(1-\frac{k}{2})}.$$

Proof. Assume this is not the case and let $G_1, \dots, G_k$ constitute a counterexample with the smallest k. Then, by Proposition 2.1, $k > 2$. Set $D = G_1 + \cdots + G_{k-1}$. Note that $D \in R_{(m-1)(1-\frac{k-1}{2})}$ by the minimality of k. Let $t \in \frac{1}{2}\mathbb{Z}$ be minimal such that $G_k \in R_t$. We would like to have $\Delta(G_k) + 1 \leq |G_k| - 2t - (m-1)$. If $t \leq \frac{1}{2}$, then we are all good since $\Delta(G_k) + 1 \leq |G_k| - m$. So, to get a contradiction, assume $t \geq 1$ and $\Delta(G_k) + 1 > |G_k| - 2t - (m-1)$. Then, by Corollary 5.3, $\omega(G_k) \leq (m-1)$. Also, by Corollary 5.2, $\alpha(G_k) \leq (m-1)$. Hence $|G_k| < r(m,m)$, contradicting the fact that $G_k \notin R_{\frac{1}{2}}$. Hence we do indeed have $\Delta(G_k) + 1 \leq |G_k| - 2t - (m-1)$.

We have

$$\begin{aligned}
\chi(D + G_k) =& \chi(D) + \chi(G_k) \\
\leq& \tfrac{1}{2}(\omega(D) + \omega(G_k) + \Delta(D) + \Delta(G_k) + 2) + (m-1)(1 - \tfrac{k-1}{2}) + t \\
=& \tfrac{1}{2}(\omega(D + G_k) + \Delta(D) + \Delta(G_k) + 2) + (m-1)(1 - \tfrac{k-1}{2}) + t \\
=& \tfrac{1}{2}(\omega(D + G_k) + \Delta(D) + |G_k| + 1 + \Delta(G_k) - |G_k| + 1) \\
&+ (m-1)(1 - \tfrac{k-1}{2}) + t \\
\leq& \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + \tfrac{1}{2}(\Delta(G_k) - |G_k| + 1) \\
&+ (m-1)(1 - \tfrac{k-1}{2}) + t \\
\leq& \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + \tfrac{1}{2}(-2t - 4 + 1) \\
&+ (m-1)(1 - \tfrac{k-1}{2}) + t \\
=& \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + (m-1)(1 - \tfrac{k}{2}).
\end{aligned}$$

Hence $G_1 + \cdots + G_k \in R_{(m-1)(1-\frac{k}{2})}$, contradicting our assumption. ∎

We can do a bit better than Lemma 5.9 in the following special case.

LEMMA 5.10. *If $k \geq 2$ and $G_1, \dots, G_k$ are noncomplete graphs, then*

$$G_1 + \cdots + G_k \in R_{1-\frac{k}{2}}.$$

Proof. Assume this is not the case and let $G_1, \dots, G_k$ constitute a counterexample with the smallest k. Then, by Proposition 2.1, $k > 2$. Set $D = G_1 + \cdots + G_{k-1}$. Note that $D \in R_{1-\frac{(k-1)}{2}}$ by the minimality of k. Let $t \in \frac{1}{2}\mathbb{Z}$ be minimal such that $G_k \in R_t$. Since $G_k \notin R_{t-\frac{1}{2}}$, if $t \geq \frac{1}{2}$, then, by Corollary 5.3, $\Delta(G_k) + 1 \leq |G_k| - 2t - 1$. If $t \leq 0$ and $\Delta(G_k) + 1 > |G_k| - 2t - 1$, then $t = 0$ and $\Delta(G_k) + 1 = |G_k|$; however, Corollary 5.1 shows that the only such graphs are complete graphs, which we have excluded, whence $\Delta(G_k) + 1 \leq |G_k| - 2t - 1$. We have

$$\begin{aligned}\chi(D + G_k) &= \chi(D) + \chi(G_k) \\ &\leq \tfrac{1}{2}(\omega(D) + \omega(G_k) + \Delta(D) + \Delta(G_k) + 2) + 1 - \tfrac{(k-1)}{2} + t \\ &= \tfrac{1}{2}(\omega(D + G_k) + \Delta(D) + \Delta(G_k) + 2) + 1 - \tfrac{(k-1)}{2} + t \\ &= \tfrac{1}{2}(\omega(D + G_k) + \Delta(D) + |G_k| + 1 + \Delta(G_k) - |G_k| + 1) + 1 - \tfrac{(k-1)}{2} + t \\ &\leq \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + \tfrac{1}{2}(\Delta(G_k) - |G_k| + 1) + 1 - \tfrac{(k-1)}{2} + t \\ &\leq \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + \tfrac{1}{2}(-2t - 2 + 1) + 1 - \tfrac{(k-1)}{2} + t \\ &= \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + 1 - \tfrac{k}{2}.\end{aligned}$$

Hence $G_1 + \cdots + G_k \in R_{1-\frac{k}{2}}$, contradicting our assumption. □

Combining Reed's conjecture with Lemma 5.9 proves the following conjecture.

CONJECTURE 5.11. *Let $m \in \mathbb{N}$. If $k \geq 2$ and $G_1, \ldots, G_k$ are graphs with $\Delta(G_i) + 1 \leq |G_i| - m$ for each i, then*

$$G_1 + \cdots + G_k \in R_{(m-1)(1-\frac{k}{2})}.$$

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