On Hitting All Maximum Cliques with an Independent Set

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Abstract: We prove that every graph G for which $\omega(G) \geq \frac{3}{4}(\Delta(G)+1)$ has an independent set I such that $\omega(G-I) < \omega(G)$. It follows that a minimum counterexample G to Reed's conjecture satisfies $\omega(G) < \frac{3}{4}(\Delta(G)+1)$ and hence also $\chi(G) > \lceil \frac{7}{6}\omega(G) \rceil$. This also applies to restrictions of Reed's conjecture to hereditary graph classes, and in particular generalizes and simplifies King, Reed and Vetta's proof of Reed's conjecture for line graphs. J Graph Theory 66: 32–37, 2010

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1. INTRODUCTION

We prove the following general lemma and apply it to Reed's conjecture.

The Main Lemma. *If G is a graph with $\omega(G) \geq \frac{3}{4}(\Delta(G)+1)$, then G has an independent set I such that $\omega(G-I) < \omega(G)$.*

In [9], Reed conjectured the following upper bound on the chromatic number.

Reed's Conjecture. *For every graph G we have $\chi(G) \leq \lceil (\omega(G)+\Delta(G)+1)/2 \rceil$.*

Observation. If we could always find an independent set whose removal decreased both ω and Δ, then the conjecture would follow by simple induction since we can give the independent set a single color and use at most $\lceil(\omega(G)+\Delta(G)+1)/2\rceil-1$ colors on what remains. Expanding the independent set given by The Main Lemma to a maximal one shows that this sort of argument goes through when $\omega\geq\frac{3}{4}(\Delta+1)$. Thus, a minimum counterexample to Reed's conjecture satisfies $\omega<\frac{3}{4}(\Delta+1)$ and hence also $\chi>\lceil\frac{7}{6}\omega\rceil$.

Reed's conjectured upper bound was proved for line graphs of multigraphs by King, Reed and Vetta in [4], for quasi-line graphs by King and Reed in [5], and recently King and Reed proved it for all claw-free graphs (see King's thesis [6]). The line graphs of multigraphs result follows from the following theorem.

Theorem D. *If G is a graph with $\chi(G)\leq\lceil\frac{7}{6}\omega(G)\rceil$ and for every proper induced subgraph H of G we have $\chi(H)\leq\lceil(\omega(H)+\Delta(H)+1)/2\rceil$, then we also have $\chi(G)\leq\lceil(\omega(G)+\Delta(G)+1)/2\rceil$.*

King, Reed and Vetta's upper bound for line graphs of multigraphs follows immediately from Theorem D, a bound of Caprara and Rizzi (see [1]) and the bound of Molloy and Reed on fractional colorings (see [8, Chapter 21, Section 3]). We write $\chi^*(G)$ for the fractional chromatic number of a graph G.

Caprara and Rizzi. *Let H be a multigraph and $G=L(H)$. Then*

$$\chi(G)\leq\max\{\lfloor 1.1\Delta(H)+0.7\rfloor,\lceil\chi^*(G)\rceil\}$$

Fractional Version. *For every graph G we have $\chi^*(G)\leq(\omega(G)+\Delta(G)+1)/2$.*

King, Reed and Vetta. *If G is the line graph of a multigraph, then $\chi(G)\leq\lceil(\omega(G)+\Delta(G)+1)/2\rceil$.*

To prove this result, they consider a minimum counterexample $G=L(H)$. This must have $\Delta(G)<\frac{3}{2}\Delta(H)-1$; otherwise, Caprara and Rizzi's result implies Reed's bound. When $\Delta(G)<\frac{3}{2}\Delta(H)-1$, they use the structure of line graphs to prove that G contains an independent set intersecting all maximum cliques. However, this second step can easily be replaced with an application of Theorem D, since $\lfloor 1.1\omega(G)+0.7\rfloor\leq\lceil\frac{7}{6}\omega(G)\rceil$. This example demonstrates the usefulness of Theorem D as a general tool, requiring no specific structural analysis to find the independent set. On the other hand, the structural analysis has the benefit of leading to a polynomial time algorithm for actually finding the desired independent set in a supposedly minimum counterexample (see [4]).

2. PROOF OF THE MAIN LEMMA

We need three lemmas. The first is due to Hajnal (see [2]).

Lemma 1. *Let G be a graph and $\mathcal{Q}$ a collection of maximum cliques in G. Then*

$$|\cap\mathcal{Q}|\geq 2\omega(G)-|\cup\mathcal{Q}|.$$

Proof. Assume (to reach a contradiction) that the lemma is false and let $\mathcal{Q}$ be a counterexample with $|\mathcal{Q}|$ minimal. Put $r=|\mathcal{Q}|$ and $\mathcal{Q}=\{Q_1,\ldots,Q_r\}$. Consider the set $W=(Q_1\cap\bigcup_{i=2}^r Q_i)\cup\bigcap_{i=2}^r Q_i$. Plainly, W is a clique. Thus

$$\begin{aligned}
\omega(G) &\ge |W|\\
&=\left|(Q_1\cap\bigcup_{i=2}^r Q_i)\cup\bigcap_{i=2}^r Q_i\right|\\
&=\left|Q_1\cap\bigcup_{i=2}^r Q_i\right|+\left|\bigcap_{i=2}^r Q_i\right|-\left|\bigcap_{i=1}^r Q_i\cap\bigcup_{i=2}^r Q_i\right|\\
&=|Q_1|+\left|\bigcup_{i=2}^r Q_i\right|-\left|\bigcup_{i=1}^r Q_i\right|+\left|\bigcap_{i=2}^r Q_i\right|-\left|\bigcap_{i=1}^r Q_i\right|\\
&=\omega(G)+\left|\bigcup_{i=2}^r Q_i\right|+\left|\bigcap_{i=2}^r Q_i\right|-\left|\bigcup_{i=1}^r Q_i\right|-\left|\bigcap_{i=1}^r Q_i\right|\\
&\ge \omega(G)+2\omega(G)-\left|\bigcup_{i=1}^r Q_i\right|-\left|\bigcap_{i=1}^r Q_i\right|,
\end{aligned}$$

where the final step follows by the minimality of $|\mathcal{Q}|$. Thus $|\bigcap_{i=1}^r Q_i|\ge 2\omega(G)-|\bigcup_{i=1}^r Q_i|$ giving a contradiction. ■

The second lemma we need is an improvement of Hajnal's result for graphs satisfying $\omega>\frac{2}{3}(\Delta+1)$ due to Kostochka (see [7]). We reproduce Kostochka's proof here to serve as an English translation.

Clique Graph. *Let G be a graph and $\mathcal{Q}$ the collection of all maximum cliques in G. The clique graph of G is the graph with vertex set $\mathcal{Q}$ and an edge between $Q_1\neq Q_2\in\mathcal{Q}$ if and only if Q_1 and Q_2 intersect. Let $\mathcal{C}(G)$ be the components of the clique graph of G.*

Lemma 2. *Let G be a graph with $\omega(G)>\frac{2}{3}(\Delta(G)+1)$. Then for every $C\in\mathcal{C}(G)$ we have*

$$|\cap V(C)|\ge 2\omega(G)-(\Delta(G)+1).$$

Proof. Assume (to reach a contradiction) that the lemma is false and let G be a counterexample with the minimum number of vertices. Then there is some component $C\in\mathcal{C}(G)$ with $|\cap V(C)|<2\omega(G)-(\Delta(G)+1)$. By minimality of G, we have $G=\bigcup V(C)$. Put $D=\cap V(C)$. Note that if $|D|\ge 1$, then $|G|\le\Delta(G)+1$ and the result follows from Lemma 1. We can therefore assume that $|D|=0$ and hence $|V(C)|\ge 3$.

By Lemma 1 we have $|G|\ge 2\omega(G)$. Put $V(C)=\{Q_1,\ldots,Q_r\}$. Take $x\in V(G)$ that is in the minimum number of the Q_i. Without loss of generality, say $x\in Q_i$ for $1\le i\le t$ for some $t\ge 1$. Consider the set

$$A=\bigcap_{i=1}^t Q_i-\bigcup_{i=t+1}^r Q_i.$$

If $y \in G-A$, then $y \notin \bigcap_{i=1}^{t} Q_i$ or $y \in \bigcup_{i=t+1}^{r} Q_i$. In the former case, we must have $y \in \bigcup_{i=t+1}^{r} Q_i$ for otherwise y would be in fewer than t of the Q_i contradicting the minimality of x. Thus $G-A \subseteq \bigcup_{i=t+1}^{r} Q_i$. Hence $G-A = \bigcup_{i=t+1}^{r} Q_i$.

To apply the minimality of G to $G-A$, all we need to show is that $G-A$ has a single clique component. Clearly, this will follow if we show that the clique graph of G is complete. Since the clique graph is connected, it will be enough to show that it is transitive, i.e. contains no induced path on three vertices. So, let Q_1, Q_2, Q_3 be distinct maximum cliques and assume that $Q_1 \cap Q_2 \neq \emptyset$ and $Q_2 \cap Q_3 \neq \emptyset$. Then $|Q_1 \cap Q_2| = |Q_1| + |Q_2| - |Q_1 \cup Q_2| \geq 2\omega(G) - (\Delta(G)+1)$. Hence

$$\begin{aligned} |Q_1 \cap Q_3| &\geq |Q_1 \cap Q_2 \cap Q_3| \\ &\geq |Q_1 \cap Q_2| - (|Q_2| - |Q_2 \cap Q_3|) \\ &\geq 2\omega(G) - (\Delta(G)+1) - (\omega(G) - (2\omega(G) - (\Delta(G)+1))) \\ &= 3\omega(G) - 2(\Delta(G)+1) > 0. \end{aligned}$$

Thus $Q_1 \cap Q_3 \neq \emptyset$, showing that the clique graph of G is transitive.

So we may apply minimality of G to conclude that $|\bigcap_{i=t+1}^{r} Q_i| \geq 2\omega(G) - (\Delta(G)+1)$. In particular, $|G-A| \leq \Delta(G)+1$. Since $A \subseteq Q_1 - Q_r$ and $Q_1 \cap Q_r \neq \emptyset$, we have $|A| \leq \omega(G) - (2\omega(G) - (\Delta(G)+1)) = \Delta(G)+1-\omega(G)$. But then

$$\begin{aligned} |G| &= |A| + |G-A| \\ &\leq \Delta(G)+1-\omega(G)+\Delta(G)+1 \\ &= 2(\Delta(G)+1) - \omega(G) \\ &< 2\omega(G). \end{aligned}$$

This contradicts the fact that $|G| \geq 2\omega(G)$. ■

Kostochka gives the example of C_5 with each vertex blown up to a k-clique to show that the $\omega > \frac{2}{3}(\Delta+1)$ condition in Lemma 2 is best possible.

The third lemma we need is a result of Haxell (see [3]) on independent transversals.

Lemma 3. *Let k be a positive integer, let H be a graph of maximum degree at most k and let $V(H) = V_1 \cup \cdots \cup V_n$ be a partition of the vertex set of H. Suppose that $|V_i| \geq 2k$ for each i. Then H has an independent set $\{v_1, \ldots, v_n\}$ where $v_i \in V_i$ for each i.*

Proof of The Main Lemma. Let G be a graph satisfying $\omega \geq \frac{3}{4}(\Delta+1)$. Put $\mathcal{C}(G) = \{C_1, \ldots, C_r\}$. By Lemma 2, the mutual intersection F_i of the maximum cliques in C_i satisfies $|F_i| \geq 2\omega(G) - (\Delta(G)+1)$ for each i. Since every vertex $v \in F_i$ is in a maximum clique in $\bigcup V(C_i)$, v is adjacent to at most $\Delta(G)+1-\omega(G) \leq \frac{1}{4}(\Delta(G)+1)$ vertices outside of $\bigcup V(C_i)$.

Let H be the graph with $V(H) = \bigcup_i V(F_i)$ and an edge between $v, w \in V(H)$ if and only if $vw \in E(G)$ and v and w are in different clique components in G. Then, by the above, $\Delta(H) \leq \Delta(G)+1-\omega(G)$.

Consider the partition $\{F_i\}_i$ of $V(H)$. We have

$$\begin{aligned}|F_i| &\geq 2\omega(G)-(\Delta(G)+1)\\ &\geq 2\tfrac{3}{4}(\Delta(G)+1)-(\Delta(G)+1)\\ &= \tfrac{1}{2}(\Delta(G)+1)\\ &\geq 2\Delta(H).\end{aligned}$$

Thus, by Lemma 3, H has an independent set $I=\{v_1,\ldots,v_n\}$ where $v_i \in F_i$ for each i. Since F_i is contained in all the maximum cliques in C_i, we have $\omega(G-I)<\omega(G)$. ■

3. PROOF OF THEOREM D

Theorem D is an easy consequence of The Main Lemma.

Proof of Theorem D. Assume (to reach a contradiction) that the theorem is false and let G be a counterexample with the minimum number of vertices. First assume that $\omega(G)\geq\frac{3}{4}(\Delta(G)+1)$. Then by The Main Lemma we have an independent set I with $\omega(G-I)<\omega(G)$. Plainly, we may assume that I is maximal (and hence $\Delta(G-I)<\Delta(G)$). Put $H=G-I$. Then, by minimality of G, we have

$$\begin{aligned}\chi(G) &\leq 1+\chi(H)\\ &\leq 1+\left\lceil\frac{\omega(H)+\Delta(H)+1}{2}\right\rceil\\ &\leq 1+\left\lceil\frac{\omega(G)-1+\Delta(G)-1+1}{2}\right\rceil\\ &\leq \left\lceil\frac{\omega(G)+\Delta(G)+1}{2}\right\rceil.\end{aligned}$$

This is a contradiction; hence, we must have $\omega(G)<\frac{3}{4}(\Delta(G)+1)$. But then

$$\begin{aligned}\left\lceil\tfrac{7}{6}\omega(G)\right\rceil &\geq \chi(G)\\ &> \left\lceil\frac{\omega(G)+\Delta(G)+1}{2}\right\rceil\\ &\geq \left\lceil\frac{\omega(G)+\frac{4}{3}\omega(G)}{2}\right\rceil\\ &= \left\lceil\tfrac{7}{6}\omega(G)\right\rceil.\end{aligned}$$

This final contradiction completes the proof. ■

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