COMPACTNESS: FROM GÖDEL TO HEINE-BOREL

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Abstract. Every beginning real analysis student learns the classic Heine-Borel theorem, that the interval $[0, 1]$ is compact. The standard proof involves techniques such as constructing a sequence and appealing to the completeness of the reals (which some may find unsatisfying). In this article, we present a different perspective by showing how the Heine-Borel theorem can be derived from a few fundamental results in mathematical logic. In particular, we put an ultrametric on the space of infinite binary sequences. Compactness of this space can be established from Brouwer's fan theorem. This result can be derived from either König's infinity lemma or from Gödel's compactness theorem in model theory. The Heine-Borel theorem is an immediate corollary. This illustrates an interesting connection between the fundamental yet different notions of compactness in analysis and compactness in logic.

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1. The Heine-Borel theorem

Think back to your first real analysis class. In the beginning, the definitions were fairly straightforward. Open and closed sets made sense, because of the common usage of open and closed intervals in previous math classes. It was a bit odd that open sets could also be closed, or that sets could be neither open nor closed, or both. But this was "higher math," so you could let that one slide (and as a bonus, you added the fantastic word "clopen" to your vocabulary!). Then, completely out of nowhere, came the definition of "compact:" A set X is *compact* if every open cover of X has a finite subcover. Why would anyone ever find themselves with an open cover, let alone try to extract a finite subcover? As you sat there in class figuring out what this entailed, the professor wrote the following two sentences on the board, with "Heine-Borel" preceding one of them.[1]

- The interval $[0, 1]$ is compact.
- A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

You might remember what came next. From an arbitrary infinite sequence contained in $[0, 1]$, a divide-and-conquer technique to construct a particular sequence of nested intervals, from this a sequence of real numbers, and then a summon to the completeness of the reals to guarantee that this sequence converges. Perhaps you found this enlightening and beautiful, perhaps the details were lost on you, or perhaps, at this early stage in your career, the appeal to the completeness of the reals felt a little unsatisfying. About this time, it might have dawned on you that your roommate (who would become a computer science major after taking said real analysis course) had a point: mathematicians make a living saying the simplest things in the most difficult round-about way!

By now, you understand in ways you never could have imagined back then, how wise your old roommate was. But you also remember what attracted you to mathematics in the first place, those mysterious qualities that, like the silver bell in the *Polar Express*, could only be heard by a select few. Your friends (old roommate included) shook their heads and exchanged private smiles when you marveled at the sheer beauty of mathematics, such as the surprising connections between seemingly unrelated topics, and the way that a basic result could be proven in vastly different ways. In fact, it is likely these reasons why you are reading this paper right now, and it is precisely these reasons that drove the authors to write it. So jump onboard, and enjoy a quick but enlightening tour of diverse topics such as ultrametrics and model theory, and we'll drop you off back in your first real analysis course at the classic Heine-Borel theorem. Along the way, we'll discover an alternative proof of the Heine-Borel theorem using simple techniques from first-order logic. We'll also see how the Gödel compactness theorem from logic implies the Heine-Borel theorem, providing a bridge between these two classic compactess theorems.

[1]Textbooks vary as to which of these statements is called the Heine-Borel theorem and which one is a lemma or corollary. We will refer to the compactness of $[0, 1]$ as the Heine-Borel theorem. See McCleary [5].

2. A Combinatorial Lemma

We begin by stating a combinatorial lemma due to Brouwer [1], and two proofs of it. These proofs are quite different; one follows from König's infinity lemma, and the other from Gödel compactness. We will briefly explain these concepts for those unfamiliar with them.

Brouwer's fan theorem. *Let $\mathcal{B}$ be a collection of finite bitstrings (binary sequences) so that every infinite bitstring has an initial segment in $\mathcal{B}$. Then there is a finite subset $\mathcal{A} \subseteq \mathcal{B}$ so that every infinite bitstring has an initial segment in $\mathcal{A}$.*

To motivate the fan theorem, consider a few easy cases. If $\mathcal{B}$ contains the bitstrings 0 and 1, then $\mathcal{A} = \{0, 1\}$ clearly works. If $\mathcal{B}$ contains 00, 01, 10, and 11, then $\mathcal{A} = \{00, 01, 10, 11\}$ works. The fan theorem says that as long as every binary sequence contains some element from $\mathcal{B}$ as an initial sequence, then we can always find a *finite* subset $\mathcal{A}$ with this property as well. Our first proof of this theorem follows from *König's infinity lemma*, which says that a countably infinite tree where every vertex has finite degree contains an infinite path [3]. (We leave the simple proof of this lemma to the curious reader.)

(The Königian Proof). Assume (to reach a contradiction) that the fan theorem is false. Recursively construct a rooted tree $T_{\mathbb{F}_2}$ with the empty bitstring at the root so that the children of b are the bitstrings $b0$ and $b1$. Now remove all bitstrings from $T_{\mathbb{F}_2}$ that have an initial segment in $\mathcal{B}$ to get the tree T.

For every $n \geq 1$ there exists a length-n bitstring with no initial segment in $\mathcal{B}$ (if every bitstring of length n had an initial segment in $\mathcal{B}$, then the bitstrings from $\mathcal{B}$ of length at most n would work for $\mathcal{A}$). Thus, T is infinite.

Since every vertex of T has finite degree, we may apply König's infinity lemma to get an infinite path through T starting at the root. Hence we have a sequence $y_1, y_2, y_3, \ldots$, where y_i is a length-i bitstring, y_i is an initial segment of y_{i+1}, and none of the y_i's have initial segments in $\mathcal{B}$. Let y be the infinite bitstring with length-i initial segment y_i for each i. If y had an initial segment of length n in $\mathcal{B}$, then y_n would have an initial segment in $\mathcal{B}$, which is forbidden by construction. Hence y has no initial segment in $\mathcal{B}$, and this contradiction completes the proof. □

3. Satisfiability of truth-functional propositional formulae.

We can get a more transparent proof of the lemma using some basic metalogical results from propositional calculus. We will start with a few simple definitions.

Given a set of Boolean variables $\{x_1, x_2, x_3, \ldots\}$, define a *formula* as follows:

(i) Every $x_i \in \{x_1, x_2, x_3, \ldots\}$ is a formula.
(ii) Every finite conjunction ($\wedge$) of formulae is a formula.
(iii) Every negation ($\neg$) of a formula is a formula.
(iv) Nothing else is a formula.

For example, the following string of symbols is a formula: $\neg(x_1 \wedge x_2) \wedge x_4 \wedge \neg x_3$. Recall that "$\wedge$" means "and" and "$\neg$" means "not." A set of formulae Γ is *satisfiable* if there is some assignment of 1 ("true") and 0 ("false") to the variables that makes every formula in Γ simultaneously true.

For example, the set

$$\{x_1\,, \quad \neg(x_1 \wedge x_2)\,, \quad \neg(\neg x_2 \wedge \neg x_3)\}$$

is satisfied by the the assignment of $x_1 = 1$, $x_2 = 0$, and $x_3 = 1$, which we write as $\{(x_1, 1), (x_2, 0), (x_3, 1)\}$. However, the set

$$\{x_1\,, \quad \neg(x_1 \wedge x_2)\,, \quad \neg(\neg x_2 \wedge \neg x_3)\,, \quad \neg x_3\}$$

is not satisfiable, because the unique assignment that mutually satisfies the first three formulae does not satisfy the fourth formula.

Now, suppose we have a (countably) infinite set of formulae. Clearly, if this set is satisfiable, then every finite subset of formluae is also satisfiable. An easy consequence of *Gödel's compactness theorem* [2] from model theory [4] says that that the converse is also true.

Compactness theorem for propositional calculus. *A set Γ of propositional formulae is satisfiable if and only if every finite subset of Γ is satisfiable.*

We can use this result to give a straightforward proof of Brouwer's fan theorem. The basic idea is to construct for each bitstring $b \in \mathcal{B}$, a formula $N(b)$ from the variables $\{x_1, x_2, x_3, ...\}$ that is not satisfied by the assignment $\{(x_1, a_1), (x_2, a_2), (x_3, a_3), \ldots\}$ if and only if b is an initial segment of $a_1a_2a_3\cdots$, and then apply the compactness theorem in a clever manner.

(The Gödelian Proof). Consider a set $\{x_1, x_2, x_3, \ldots\}$ of propositional variables. Let K be the following set of formulae:

$$\{N(b) \mid b \in \mathcal{B}\}\ ,$$

where for all $b = b_1b_2\cdots b_k \in \mathcal{B}$, $N(b) = \neg\,[\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_k]$ and for all b_i of each $b_1b_2\cdots b_k$,

$$\phi_i = \begin{cases} x_i, & \text{if } b_i = 1 \\ \neg x_i, & \text{if } b_i = 0 \end{cases}$$

By construction, $N(b)$ is satisfied by the assignment $\{(x_1, a_1), (x_2, a_2), (x_3, a_3), \ldots\}$ if and only if b is not an initial segment of $a_1a_2a_3\cdots$. For example, if $b = b_1b_2b_3 = 100$, then

$$N(b) = \neg[x_1 \wedge \neg x_2 \wedge \neg x_3]$$

and this formula is satisfied by the assignment $\{(x_1, a_1), (x_2, a_2), (x_3, a_3), \ldots\}$ as long as $a_1a_2a_3 \neq 100$.

Now, assume (to reach a contradiction) that the fan theorem is false. Then, for any finite $\mathcal{A} \subseteq \mathcal{B}$, there exists a bitstring $a_1a_2a_3\cdots$ that has no initial segment in $\mathcal{A}$. Hence, every finite subset of formulae of K is satisfiable, and by compactness, K is satisfiable. But by construction, this yields an infinite bitstring $a_1a_2a_3\cdots$ with no initial segment in $\mathcal{B}$. This contradiction completes the proof. □

4. The Bit-Metric

Equipped with the fan theorem, we resume our tour in the land of bitstrings. Let $\mathbb{F}_2 = \{0, 1\}$, and let $\mathbb{F}_2^{\mathbb{N}}$ denote the set of infinite bitstrings. Again, we write a bitstring as $a = a_1 a_2 \cdots$, and call the individual a_i's *bits*. Define the function ι that sends an element of $\mathbb{F}_2^{\mathbb{N}}$ to the corresponding number in $[0, 1]$ written in binary, by

$$\iota\colon \mathbb{F}_2^{\mathbb{N}} \longrightarrow [0,1] , \qquad \iota(a_1 a_2 a_3 \cdots) = \sum_{i=1}^{\infty} a_i 2^{-i} = 0.a_1 a_2 a_3 \ldots .$$

At this point, we must be careful not to overlook the fact that binary decimals have a few subtle pesky properties, such as the fact that

$$\iota(a_1 a_2 \cdots a_k 1000 \cdots) = \iota(a_1 a_2 \cdots a_k 0111 \cdots) .$$

Fortunately, ι is injective on bitstrings not of this form. With this in mind, we say that a binary decimal representation of $x \in [0, 1)$ is in *standard form* if there are infinitely many 0s, and we say that $0.1111 \cdots$ is the standard form of 1. When we speak of a number $x \in [0, 1]$, we shall assume that it is written in standard binary form. With this assumption, we can define $\iota^{-1}(x)$ to be the preimage of $x \in [0, 1)$ that has infinitely many 0s (and $\iota^{-1}(1) = 11111\ldots$).

Next, we put a metric on $\mathbb{F}_2^{\mathbb{N}}$ by saying that two distinct bitstrings a and b are a distance $\beta(a, b) = 2^{-k}$ apart, where k is the last bit at which a and b agree. It is straightforward to show that $(\mathbb{F}_2^{\mathbb{N}}, \beta)$ is an *ultrametric*, and we call it the *bit-metric* on $\mathbb{F}_2^{\mathbb{N}}$. An ultrametric is any metric that satisfies the strong triangle inequality:

$$\beta(a, c) \le \max\{\beta(a, b), \beta(b, c)\} ,$$

and this gives it some extra special properties such as:

- *Russian Nested Doll property of balls*: If $B_r(a) \cap B_r(b) \neq \emptyset$, then either $B_r(a) \subseteq B_r(b)$ or $B_r(a) \supseteq B_r(b)$.
- *Center of the universe property*: If $|a - b| < r$, then $B_r(a) = B_r(b)$ (i.e., every interior point of a radius-r ball can be taken to be the center).

These properties are very useful when studying $(\mathbb{F}_2^{\mathbb{N}}, \beta)$, and we utilize them in papers that are much more difficult to read than this one. However, we will not need them for Heine-Borel, but we mention them for completeness (of the paper, not the reals).

At this point, you might be suspecting that the map ι, being so simple, is continuous. This is indeed correct since, by definition, $|\iota(a) - \iota(b)| \le \beta(a, b)$ for any $a, b \in \mathbb{F}_2^{\mathbb{N}}$.

Lemma 1. *The map ι is continuous under the bit-metric.*

However, what may come as a surprise is that under the bit-metric, $\mathbb{F}_2^{\mathbb{N}}$, a collection of infinite sequences, is compact.

Lemma 2. $(\mathbb{F}_2^{\mathbb{N}}, \beta)$ *is compact.*

Proof. Consider an open cover $\cup_{i \in I} B_{\epsilon_i}(a_i) = \mathbb{F}_2^{\mathbb{N}}$ of balls, where each $a_i \in \mathbb{F}_2^{\mathbb{N}}$. The ball $B_{\epsilon_i}(a_i)$ contains precisely the bitstrings that agree with the binary decimal form of a_i on at least the first $k_i := \lfloor \log_2(\epsilon_i^{-1}) \rfloor$ bits. Let S_i be this initial segment of a_i, which is a length-k_i binary sequence. Now, a bitstring $b \in \mathbb{F}_2^{\mathbb{N}}$ is in $B_{\epsilon_i}(a_i)$ if and only if S_i is an initial segment of b. Consider the set $\mathcal{B} = \{S_i \mid i \in I\}$ of finite bitstrings. Since every infinite bitstring lies in some ball $B_{\epsilon_i}(a_i)$, every infinite bitstring in $\mathbb{F}_2^n$ has an initial segment S_i from $\mathcal{B}$. Brouwer's fan theorem gives us a finite set of initial segments $\mathcal{A} \subseteq \mathcal{B}$ for $\mathbb{F}_2^n$. This means that every bitstring in $\mathbb{F}_2^n$ lies in the union

$$\bigcup_{a_i \in \mathcal{A}} B_{\epsilon_i}(a_i) = \mathbb{F}_2^{\mathbb{N}} ,$$

and thus we have found a finite subcover of $\mathbb{F}_2^{\mathbb{N}}$. □

Equipped with Lemmas 1 and 2, we can now present The Shortest Proof of Heine-Borel Ever.

Theorem 3 (Heine-Borel)**.** *The interval* $[0, 1]$ *is compact.*

Proof. $\iota(\mathbb{F}_2^{\mathbb{N}}) = [0, 1]$ is the continuous image of a compact set. □

This concludes our tour, now that we have arrived back at your first real analysis class, on that special day when you first saw the Heine-Borel theorem proven. Though we aren't suggesting that this proof should replace the standard divide-and-conquer technique, we hope that you find the strategy presented here intriguing. For those of you out there that have yet to take real analysis, but are advanced and motivated enough to be reading this article, pay attention. When you find yourself in an analysis class, and the professor draws that little box at the end of the proof of Heine-Borel, raise your hand, and inquire:

"*Doesn't that just follow from König's infinity lemma, and the standard ultrametric on the space of binary sequences*?"

Then turn around and smile at your roommate.

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