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A COMPUTERIZED SYSTEM FOR GRAPH THEORY, ILLUSTRATED BY PARTIAL PROOFS FOR GRAPH-COLORING PROBLEMS[1]

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Abstract

The software system KBGRAPH, which supports graph theoretical proofs and the analysis of graph classes, is presented by developing partial proofs for two graph coloring problems. It is shown that Reed’s Conjecture, which concerns an upper bound on the chromatic number, holds for some special classes of graphs; future approaches are briefly outlined. Another strengthening of Brooks’ well-known upper bound is sketched. Details about the internal derivation strategies of the program and tools offered to the users are presented, as far as needed for an understanding of the subsequent sketch of a problem solving process. This article is written for a two-fold readership: readers who want a quick overview of the knowledge based system will find this in sections 1 – 4; for readers interested in more details of the software system, additional hints on its implementation, technical data, and the availability of the program are compiled in the last section.

1. A First Look at Open Problems

The knowledge-based system *KBGRAPH* is destined to support graph theoretical proofs and the analysis of graph classes. Instead of a boring sequence, describing one function of the program after the other, we want to let the reader participate in a step-by-step search for subsequent improvements, aiming at a proof of *Reed’s Conjecture*. To date only partial proofs have been found. Typically, in the course of such a stepwise search, new relations between graph invariants are discovered, that are valid for all graphs and hence of independent interest.

Reed’s Conjecture is an extension of a well-known upper bound on the *chromatic number*, $\chi(G)$, **[1]**.

Conjecture (Reed): For any graph *G*,

(1) $$\chi(G) \le \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil,$$

where $\Delta(G)$ denotes the maximum degree and $\omega(G)$ is the clique number of *G*. □

The chromatic number χ is a graph invariant that is connected with a great variety of other invariants, such that sharper bounds for χ may lead to an improved knowledge about other variables.

The following sketch shows how structural knowledge on the one hand, and inequalities stored within the knowledge based system, on the other hand, can be used to proceed toward partial proofs. The system *KBGRAPH* will frequently be “in the background”. In any case, the following text is written such that no prior knowledge about the *KBGRAPH* system (nor of other such systems) is needed. Therefore, only a necessary short overview is given in the beginning (Section 3). Additional details for interested readers are placed in a separate section (Section 5).

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[1] This paper is a revised and extended version of an article that appeared in *MATCH Commun. Math. Comput. Chem.*, **58**, 445–480 (2007), with kind permission.

As a by-product, new inequalities, partially connected with Reed's Conjecture, are listed separately (Section 4.5); in particular, sufficient conditions for a strengthening of Brooks' inequality such that $\chi \leq \Delta$ is replaced by $\chi \leq \Delta - 1$.

2. Notation

In this paper, *graph coloring* refers to coloring the vertices of graphs. All graphs considered are simple (finite, undirected, with no loop or multiple edge). As usual, G is a graph with p vertices and q edges, and $\overline{G}$ is the complement of G. Special graph classes are the *complete graphs* K_m, the *cycle graphs* C_m, and the *claw* $K_{1,3}$. Frequently occurring variables are the *minimum degree* δ and the *maximum degree* Δ, the *clique number* ω (that is, the number of vertices of the greatest clique), the *independence number* β_0, and the *vertex-cover number* α_0, the *vertex connectivity* κ, and the *edge connectivity* κ_1. We also write χ instead of $\chi(G)$, and so forth if this does not result in ambiguity.

3. Overview of the System ***KBGRAPH***

The knowledge-based system *KBGRAPH* has two objectives:

- to analyze given classes of graphs, and
- to support proofs of graph theoretical hypotheses.

At present, the knowledge base consists of about 1,700 entries. Each such entry has the form of a known property of a graph invariant (e.g., $\omega \geq 2$) or of a relation between graph invariants, which may be unconditional (e.g., $\chi \geq \omega$) or conditional (*if…, then…*). Integer, real, and Boolean variables are permitted, as well as logical connectives (*and*, *or*, *not*). Each entry (except for some trivial cases) is equipped with a reference.

About 50 graph invariants are implemented, including all of the invariants mentioned in this paper. Material on other variables has been accumulated in the paper form, but not yet entered into the knowledge base.

A problem description consists of a finite list of *user-defined restrictions*. These have the same form as the knowledge-base entries: conditional or unconditional equations or inequalities; in practice, unconditional statements are more frequent. Any property of the considered class of graphs that is already known can be entered here as a user-defined restriction (see the example in Section 4.2). In those cases where an attempt of a mathematical proof is made, a problem description lists known properties of a hypothetical counterexample.

At the onset of the evaluation process, just after reading the user defined restrictions, an internal duplicate of the knowledge base is generated, and this is confronted with the user defined restrictions. The main evaluation process works in the usual mathematical style (forward chaining): known numerical and Boolean values are inserted, and the formulæ are simplified. In this way, the temporary duplicate of the knowledge base is permanently updated. As soon as the *if* part of a conditional statement is found to be true, this *if* part is deleted and the *then* part remains as an unconditional statement. If a *then* part turns out to be false, the negation of the *if* part is retained as a true statement. Transitivity of equality and of inequality relations is taken into account (e.g., $\kappa \leq \kappa_1$ and $\kappa_1 \leq \delta$ implies $\kappa \leq \delta$). A special table is set up and permanently updated; this stores the currently best known numerical values for the lower and upper bounds of the numerical variables.

Within the general framework of forward chaining, a selection of specific techniques applied within the inference process can be sketched here:

- rounding in the case of integer variables: For example, $\chi < 9/2$ is replaced by $\chi \leq 4$;
- deletion of formulæ that are inferior to other entries in the set of transformed formulæ;
- conclusions derived from the monotonicity of arithmetic functions: If, for example, $y = f(x)$ is a monotonically increasing function for $a \leq x \leq b$ ($a < b$), then it can be derived that $f(a) < f(b)$. (Monotonicity of a function can be recognized for linear or quadratic expressions and for functions of the type $y = c\log(x) + d$.)

Additional special methods of evaluation are described in Section 5.

When an inference run has ended, then generally concrete values for some variables have been identified, and improved bounds to some numerical variables have been obtained. These values and bounds are displayed to the user. Another part of the intermediate results consists of those knowledge-base entries that were altered by the evaluation runs. These formulæ can optionally be displayed on a screen, either completely or in a selective manner by use of a retrieval function. After any evaluation run, the user may enter further knowledge—possibly triggered by studying the recent results—and restart the evaluation. This may be repeated recursively as long as an improvement is found by the system.

If, in the case of an attempted proof, a contradiction is found, this means that the underlying class of graphs—that is, the class of hypothetical counterexamples—is empty, or, equivalently, that the hypothesis has been proved. This is signalled to the user, together with data about the formulæ that led to that contradiction.

KBGRAPH is consequently organized for interactive working. After the end of an evaluation run the user has the chance to:

- enter additional knowledge in the same style as the initial user defined restrictions;
- edit a single formula (e.g., to simplify an arithmetic expression by hand);
- enter the knowledge that some *if* part is true or some *then* part is false;
- tentatively insert numerical values for a numerical variable (in the case of a contradiction it is possible to increase a lower or to reduce an upper bound).

For each result, a *derivation tree* can be displayed, which shows how the result has been derived. The formula numbers lead to references from which formulæ of the original knowledge base were taken.

Some advanced evaluation techniques are also implemented; for example, working with *case distinctions*. For Boolean variables, the two alternatives can analyzed separately (e.g., *regular/not regular*); in the case of a numerical variable, the domain is decomposed into partial intervals (see an example in Section 4.2). There may be an identical improvement for the various cases that were detected in quite distinct methods of derivation. (For additional advanced evaluation techniques see Sections 4.2 and 5.2.)

A characteristic phenomenon appearing in evaluation processes can be dubbed *knowledge propagation*: improved knowledge about one variable is likely to advance the knowledge about other variables. Inspection of the ways in which surprising results come up suggests the term *crossword-puzzle phenomenon*. When a crossword puzzle is solved, a single new finding can trigger a *chain reaction* of additional new findings, such that finally entries for distant places will be found. Hence, any increase in a lower or decrease in an upper bound can be considered a chance for more progress. Furthermore, conditional formulas are activated as soon as a bound in a condition is reached.

The system supplies improved knowledge about the class of graphs considered. In particular, exact values for some graph invariants, sharper bounds for most other variables, and restrictions in the form of equations or inequalities to be fulfilled by graph invariants. If no proof is derived (which is the regular case for long-standing graph theoretical conjectures), then the new knowledge about properties of a counterexample may simplify the remaining task.

4. Reed's Conjecture

4.1. Problem Statement

Brooks **[1]** proved that for all connected graphs the inequality

(2) $\chi \le \Delta + 1$

holds, with equality if and only if G is a complete graph or an odd cycle (here only the case of connected graphs with $\chi \le \Delta$ is of interest). This was strengthened by Reed **[2][3]**, whose conjecture is stated above as **(1)**. Neither a proof nor a counterexample are known. This conjecture is trivial for $\omega = \Delta$ and for $\omega = \Delta + 1$. Reed **[2]** gave a proof for graphs with maximum degree $\Delta = p - 1$. A proof for all *line graphs* was presented at a conference in Berlin (June 2005, **[4]**); this proof stood in the context of a harder claim on multiedge graphs. A quick proof for line graphs—restricted to simple graphs—is obtained below as a by-product (Section 4.2, step 2). Contributions in two recent papers **[5][6]** are compiled below (Section 4.3).

4.2. A First Attempt with Reed's Conjecture

Unless otherwise stated, G denotes a counterexample to Reed's Conjecture. It is our goal to find additional and sharper constraints that the class of counterexamples need to fulfil.

Step 1: We can restrict our study to *color critical graphs* with chromatic number χ (χ-critical graphs). Every graph with chromatic number χ contains a χ-critical subgraph with the same number of vertices. If such a graph obeys **(1)**, then in any other graph generated from it by inserting edges, Δ and ω remain constant or increase, such that **(1)** continues to remain valid. Thus, we can make use of the known properties of color critical graphs. (The property *color critical* cannot be found automatically, but it is implemented in the system as a Boolean variable and is used if stated by the user.)

Step 2: By inserting $\omega = \chi$ and $\omega = \chi - 1$ into **(1)**, it turns out that **(1)** is fulfilled for these values. Hence, G must satisfy the constraint

(3) $\chi \geq \omega + 2$.

Some consequences of this property are stored in the system. Furthermore, it is known **[7]** that graphs obeying **(3)** must contain $K_{1,3}$ and/or K_5–e as an induced subgraph. These graphs are forbidden induced subgraphs for line graphs, and so it is quickly proved that **(1)** holds for line graphs.

Here, the restriction **(3)** was found by the user. In principle, it would be possible to start a program run without entering **(3)** as a user defined restriction—the program would be able to exclude $\omega = \chi$ and $\omega = \chi - 1$ in later phases. However, by doing so, the program run would be longer, and the intended demonstration would be rather clumsy. Furthermore, it should also be shown that the user's additional prior knowledge can be entered here.

Step 3: Next we check if $\chi = \omega + 2$ or $\chi = \omega - 2$ is possible. Since counterexamples are studied we have

$$\chi > \lceil (\Delta + \omega + 1)/2 \rceil$$

or equivalently

$$\chi > ((\Delta + \omega + 1 + \varepsilon)/2), \text{ and } \chi > \Delta - 1 + \varepsilon ,$$

with $\varepsilon = 1$ if $\Delta + \omega$ is even and $\varepsilon = 0$ if $\Delta + \omega$ is odd. The case $\varepsilon = 1$ must be excluded since here $\chi \leq \Delta$. Only the case $\varepsilon = 0$ remains. Then $\chi = \Delta$, and $\chi = \omega + 2$ implies $\chi \equiv \omega \pmod 2$, $\Delta \equiv \omega \pmod 2$, $\Delta + \omega \equiv 0 \pmod 2$, $\varepsilon = 1$, contrary to $\varepsilon = 0$. Hence, $\chi = \omega + 2$ is excluded, and with **(3)** we obtain

(4) $\chi \geq \omega + 3$.

This derivation cannot be accomplished by the system.

Step 4: As a next step we can compile the user defined restrictions:

R1: $\chi > \lceil (\Delta + \omega + 1)/2 \rceil$
R2: $\Delta \leq p - 2$
R3: color critical
R4: $\chi \geq \omega + 3$.

Here R1 is the negation of **(1)** since we are looking for a counterexample. R2 is a consequence of Reed's additional restriction as cited above. R3 was explained previously (Step 1), and R4 goes back to Step 3.

Step 5: With these user defined restrictions, a first evaluation run is started. Among the results only two points are worth reporting: A counterexample G is not completely multipartite, and it has $p \geq 11$. The latter finding is mainly due to a theorem by Nenov **[8]**: here, $\chi \geq 5$, and for $\omega \leq 3$ and $p \leq 10$ it follows that $\chi \leq 4$; for $\omega \geq 4$ with $\chi \geq 7$ the derivation is different.

Step 6: After the end of the first standard evaluation run, the special evaluation technique *working with case distinctions* is activated. We consider the complete case distinction {ω = 2, 3, 4, ≥ 5}, which means that the program will consecutively (but independently) handle the four cases:

$\omega = 2$, $\omega = 3$, $\omega = 4$, and $\omega \geq 5$.

A selection of the results obtained is provided in the following table:

ω	= 2	= 3	= 4	≥ 5
$p \geq$	22	12	13	15
$q \geq$	47	34	43	58
$\chi \geq$	5	6	7	8
$\gamma \geq$	2	1	2	2

where γ is the *orientable genus*. First, note a little improvement from $p \geq 11$ to $p \geq 12$. The lower bound $p \geq 22$ for $\omega = 2$ can be traced back to a result by Jensen and Royle **[9]**: a graph with $\omega = 2$ and $\chi = 5$ has $p \geq 22$ (this lower bound also holds for $\chi \geq 6$).For the lower bounds to q, there are quite a lot of inequalities in the knowledge base, some of which also use parameters like ω and Δ; one of the most efficient lower bounds for q in the case of color critical graphs was found by Kostochka *et al.* (**[10]**, see Section 4.5, nr. 2). The lower bounds for χ follow from **(4)**.

In three of the four cases we have $\gamma \geq 2$. Before we proceed to the case with $\gamma \geq 1$, we study the functioning of the program module *working with case distinctions*. The same result, $\gamma \geq 2$, was derived in three different ways for the three cases (at the same time, it was noticed here that after a program run the derivation of each result was displayed). In this particular case, we could identify the underlying knowledge-base entries:

Case 1: $\omega = 2$

If $\gamma \leq 1$ and $\omega = 2$, then $\chi \leq 4$. **[11]**

Case 3: $\omega = 4$

If $\chi \geq 7$ and $\omega \leq 6$, then $\gamma \geq 2$. **[12]**

Case 4: $\omega \geq 5$

If $\gamma \leq 1$, then $\chi \leq 7$. **[11]**

Step 7: In view of the preliminary lower bounds for γ, the user may decide to handle the troublemaker—Case 2, with $\gamma \geq 1$ —separately. To this purpose, a new program run was started for Case 2 with the *hypothesis* $\gamma = 1$ as a new user defined restriction. This program run used all known results for this case and all restrictions defined previously, in particular, $\omega = 3$ and $\chi \geq 6$. Mainly on the basis of a formula by Dirac **[13]**, the program supplies $\chi = \delta = \Delta = 6$, such that G would be regular. However, according to Gould (**[14]**, p. 247), a color critical graph with $\delta \geq 3$ and $\delta = \chi$ cannot be regular. This contradiction, which is displayed to the user, excludes $\gamma = 1$, and so $\gamma \geq 2$ has been proved for this case.

To illustrate the flexibility of the system, we show an alternative proof for $\gamma \geq 2$ in Case 2: A *semi-automatic*, computer-assisted proof, which starts from the known facts $\omega = 3$ and $\chi \geq 6$. The retrieval function is activated and as a response to the query "χ and γ", about 20 formulæ containing χ and γ are displayed on the screen. Stimulated by a formula due to Thomassen **[12]**, the user can look up the original printed version. According to this source, most of the graphs with $\gamma = 1$ have $\chi \leq 5$, and hence can be ignored here. Two exceptional graphs have $\omega \geq 4$, contrary to $\omega = 3$. For the third of Thomassen's exceptional cases, one can combine the fact that here $p \geq 12$ with findings by Albertson and Hutchinson **[15]** resulting in the consequence that this last exceptional graph can also be omitted. Thus, we derive that G has $\gamma \geq 2$, or, in other words, that **(1)** holds for planar graphs and for toroidal graphs.

4.3. Contributions from the Theory of Graph Associations

The following inequalities, valid for all graphs, are taken from two recent papers **[5][6]**. Using the concept of graph associations, a theorem is found that permits us to derive new bounds for χ by choosing special types of induced subgraphs.

Definition: Given a graph G and non-adjacent vertices a and b, we write $G/[a, b]$ for the graph obtained from G by associating (i.e., identifying) a and b into a single vertex $[a, b]$ and discarding any multiple edges. ■

Theorem 1: Let G be a graph. Then, for any induced subgraph H of G

(5) $$\chi(G) \leq \chi(H) + \frac{p(G) + \omega(G) - p(H) - 1}{2}.$$ ■

There are two immediate applications. If G is connected and if the subgraph H is identified with a longest induced path P_m of G ($m \geq 3$, such that the diameter $d(P_m) = d(G) = m - 1 \geq 2$, then **(5)** leads to

(6) $$\chi(G) \leq \frac{p(G) + \omega(G) - d(G) + 2}{2}.$$

Next, suppose that G has $g \geq 5$ (where the *girth* g is the length of a shortest cycle) and take for H a subgraph induced by a shortest cycle together with its neighborhood, then

(7) $$\chi(G) \le \frac{p(G) - g(G)(\delta(G) - 1) + 7}{2}.$$

Theorem 2: Let G be a graph. Then

(8) $$\chi(G) \le \frac{p(G) + \omega(G) - \beta_0(G) + 1}{2},$$

or equivalently (with Gallai's relation $\alpha_0 + \beta_0 = p$)

(9) $$\chi(G) \le \frac{\omega(G) + \alpha_0(G) + 1}{2}.$$ ■

For triangle-free graphs it follows that

(10) $$\chi(G) \le \frac{p(G) - \Delta(G) + 3}{2}.$$

The following three inequalities are related to Reed's Conjecture. It was found that **(1)** holds for *decomposable graphs*; that is, for graphs G with a disconnected complement $\overline{G}$, such that G can be written as a *direct sum* $G = A + B$ (where $A + B$ means that each vertex of A is adjacent to each vertex of B). If G is a counterexample to **(1)**, then $\overline{G}$ has a perfect matching if p is even. For odd p, $\overline{G}$ is nearly matching-covered; that is, $\overline{G} - v$ has a perfect matching for any vertex v (see also **[16]**). Furthermore, $\overline{G}$ is bridgeless; that is, $\kappa_1(\overline{G}) \ge 2$.

Any counterexample to **(1)** must satisfy the inequalities

(11) $$\chi(G) \le \left\lceil \frac{p(G)}{2} \right\rceil,$$

(12) $$\Delta(G) \le p(G) - \sqrt{p(G) + 2\beta_0(G) + 1},$$

(13) $$\beta_0(G) \ge 3.$$

4.4. Example of an Advanced Evaluation Technique

As remarked previously, a proof for Reed's Conjecture has not yet been found. By experimenting with the program (and material from literature) we suggest that there are two graph classes for which a solution may be relatively easy: *Triangle free graphs* and *claw free graphs*. The following account is to show—based on an example—one of the advanced features of the program that the user can apply following a usual inference run.

Triangle free graphs are characterized by $\omega = 2$. Following an ordinary program run (for counterexamples to Reed's Conjecture), then including the new constraint $\omega = 2$, a run with a case distinction (*cf.* Section 4.2, Step 6) was used. The case distinction $\{g = 4, g \ge 5\}$ led to the result that $\gamma \ge 2$ for $g = 4$ and $\gamma \ge 3$ for $g \ge 5$. This suggests a test of whether $\gamma \ge 3$ could be proved for $g = 4$ also. The output of a new program run with the constraints $g = 4$ and $\gamma = 2$ consists of fixed values for nine numerical variables (e.g., $\chi = \Delta = 5$), whereas for all other numerical variables both lower and upper bounds are supplied.

Consequently, this is an ideal candidate for a program function called *automatic insertion*: For each integer variable that is constrained from both sides, all admissible values are inserted into the formulæ of the knowledge base and a contradiction close to a bound leads to an increase of a lower bound or a decrease of an upper bound (an extension to intervals bounded at one side and to real variables can only be mentioned here, see also Section 5.2). In the concrete case, for example, the interval $22 \le p \le 86$ was replaced by $22 \le p \le 56$, and $47 \le q \le 176$ was converted into $47 \le q \le 116$. In a similar way, inclusions for most of the other integer variables were strengthened.

Other advanced techniques of evaluation were also applied to Reed's Conjecture, but in this special case they did not lead to significant progress. Therefore, these techniques are handled in a general form in Section 5.2; a summary of partial results follows in Section 4.6.

4.5. Miscellaneous Inequalities

Related inequalities, vastly scattered in literature, should be registered here (with a short derivation or reference). The following formulæ (in nr. 1–3) were found by the retrieval function of *KBGRAPH* with the query

"χ and Δ". Some of them have immediate consequences for Reed's Conjecture. (The references are supplied by the system; a selection and some editing for the sake of easy reading were required.)

1. Brooks' well-known result, that all connected graphs (except for complete graphs and odd cycles) satisfy $\chi \le \Delta$, suggests to ask for conditions under which the stronger property

(14) $\chi \le \Delta - 1$

will hold. Some of the sufficient conditions are:

(a) If $\omega = 2$, then $\chi \le 2(\Delta + 2)/3 + 1$. **[17]**
$\rightarrow$ $\omega = 2$ and $\Delta \ge 8$ implies **(14)**.

(b) If $\omega \le 3$, then $\chi \le 3(\Delta + 2)/4$. **[18]**
$\rightarrow$ $\omega \le 3$ and $\Delta \ge 7$ implies **(14)**.

(c) If $\Delta \ge 7$ and $\omega \le (\Delta - 1)/2$, then $\chi \le \Delta - 1$. **[18]**

(d) If $\chi \ge \omega + 1$ and $\Delta > (p + 1)/2$ then $\chi \le \Delta - 1$.
If $\chi \ge \omega + 1$ and $\Delta \ge 9$ and $\Delta > p/2$, then $\chi \le \Delta - 1$. **[19]**

(e) If G has no C_4 (induced subgraph or not), then $\chi \le 2\Delta/3 + 2$. **[20]**
$\rightarrow$ If G has no C_4, then $\Delta \ge 7$ implies **(14)**.

(f) If G is color critical and $\kappa = 2$ with a cutset $\{u, v\}$,
then u and v are non-adjacent, and $\Delta \ge (3\chi - 5)/2$. **[14]** (p. 227)
$\rightarrow$ Under these conditions $\Delta \ge 6$ implies **(14)**.

(g) The Borodin–Kostochka Conjecture **[18]** claims:
If $\Delta \ge 9$ and $\omega \le \Delta - 1$, then $\chi \le \Delta - 1$.
Partial proofs were compiled in **[3]**. Recently this conjecture was proved for graphs containing a doubly critical edge; that is, an edge whose removal decreases χ by 2 **[21]**.

2. For color critical graphs, given χ and p, a good lower bound for q is required. At present, the best such bound is supplied by **[10]**, under the reservation that two classes of exceptional graphs are excluded. This restriction can be by expressed by the three *or*-connected properties:

If G is color critical and $4 \le \chi \le p - 2$ and ($2\chi \ne p + 1$ or $\beta_0 \ge 3$ or $\omega < (p - 1)/2$),
then $q \ge p(\chi - 1)/2 + \chi - 3$.

3. If G contains neither C_4 nor $2K_2$ as induced subgraphs, then $\chi(G) + \chi(\overline{G}) \ge p(G)$ and $\chi(G) \le \omega(G) + 1$ **[22]**. Thus, due to **(3)**, **(1)** also holds for this special class of graphs.

4. Every counterexample to **(1)** has $\Theta_0(G) \ge 5$, where $\Theta_0(G) = \chi(\overline{G})$ is the clique-to-vertex covering number. From **[23]** (Theorems 1 and 2) it follows that for all graphs

$\omega = 2$ implies $\Theta_0 \ge \chi$,

$\omega = 3$ implies $\Theta_0 \ge \chi - 1$.

From **(3)** we have $\chi \ge \omega + 3$, and in both cases $\Theta_0 \ge 5$ follows. For the case $\omega \ge 4$ and $\chi \ge 7$ a structure theorem by Dirac **[24]** can be used: Such a graph contains two vertices with $\chi - 1$ paths between them, where no two of these paths have an edge in common. Hence, a χ-critical graph ($\chi \ge 7$) cannot be covered by less than five cliques, with the order of these cliques bounded by $\chi - 3$. Here, at least five cliques are needed for a covering and the claim is proved.

5. As is shown above, Reed's Conjecture holds for all *line graphs*. This can be extended to a broader class of graphs. Line graphs have $\lambda_p \ge -2$ (where λ_p is the smallest adjacency eigenvalue) and there are exactly two classes of connected graphs sharing this property:

- generalized line graphs
- exceptional graphs.

For a definition of a *generalized line graph*, see for example **[25][26]**. Their relevant properties can best be found through a structural characterization given by Cvetkovic **[25]** (Theorem 2.2). A *generalized cocktail party graph* is a graph isomorphic with a clique with independent edges removed. The complement of a generalized cocktail party graph consists of isolated edges and vertices and so a generalized cocktail party graph

H has $\chi(H) = \omega(H)$. If G is a generalized line graph then its edges can be partitioned into generalized cocktail party graphs such that

Each vertex is in at most two generalized cocktail party graphs, and
two generalized cocktail party graphs have at most one common vertex.

Therefore, generalized line graphs do not obey **(4)**, and can be dropped here. Exceptional graphs are defined as connected graphs with $\lambda_p \geq -2$ that are neither line graphs nor generalized line graphs. With the explicit restriction to $\lambda_p > -2$ we find that there are 573 such graphs; they have at most 8 vertices **[26]** and, hence, can be ignored here. A counterexample to Reed's Conjecture has $\lambda_p \leq -2$. (The case of equality cannot be attacked with the present tools.)

4.6. Summary of Partial Results

Reed's Conjecture was proved for the following classes of graphs:

- Line graphs, generalized line graphs, and those exceptional graphs with $\lambda_p > -2$.
- Graphs with $\chi \leq \omega + 2$.
- Planar and toroidal graphs.
- Decomposable graphs.
- $\{C_4, 2K_2\}$-free graphs.

In any counterexample, variables have to satisfy the following lower bounds (a small selection): $p \geq 12$, $q \geq 34$, $\chi \geq 5$, $\delta \geq 4$, $\Delta \geq 5$, $\beta_0 \geq 3$, $\alpha_0 \geq 8$, $\gamma \geq 2$, $\Theta_0 \geq 5$, and $\lambda_p \leq -2$. Necessary structural properties of the complement $\overline{G}$ are compiled in Section 4.3; bounds on its invariants include: $\omega(\overline{G}) \geq 3$, $\chi(\overline{G}) \geq 5$, and $\Theta_0(\overline{G}) \geq 5$.

5. Additional Details about the System *KBGRAPH*

5.1. Starting Point and General Properties

The project *KBGRAPH* was started in 1985, stimulated by the appearance of a series of papers by Brigham and Dutton **[27]**–**[30]**. Their two compilations of relations between graph invariants **[29][30]** with overall 458 entries remain unparalleled; they formed the core of the knowledge base in the first version of *KBGRAPH*. The system has been independently developed further. Whereas forward chaining (see Section 3) has been maintained as the central evaluation strategy, *KBGRAPH* is now characterized by a quantitative increase (number of graph invariants and size of the knowledge bases) and by a series of novel features, mainly related to

- the user interface and the options for flexible post-processing,
- the advanced evaluation techniques (Section 5.2),
- the options for an external control of the inference process (Section 5.3).

At present, 51 graph invariants are implemented. The three knowledge bases include about 2,100 entries: About 1,700 in the *main knowledge base* and the rest in two *auxiliary knowledge bases* required for one of the special evaluation techniques (Section 5.2). According to individual requirements, graph invariants can be newly defined, cancelled, or renamed. The knowledge-base is permanently updated: Adding, deleting, or altering of entries is possible. From time to time a single entry is replaced by a stronger version.

Based upon forward chaining as the central inference method, the inference mechanism was programmed *ad hoc*, to adapt to the specific requirements of working with formulæ (no foreign software was used). Options for an external control of the inference are outlined in Section 5.3.

After the end of an inference run, the user can enter additional knowledge and start the inference process again, or apply one or the other of the *advanced evaluation techniques* (Section 5.2). This can be done repeatedly for as long as some progress is expected. After the end of each inference run, it is possible to display a derivation tree for each single result, and, in the case where more than one derivation method led to the same result, this fact is also disclosed to the user. Thus all findings can be checked and rewritten in the usual mathematical style.

5.2. Advanced Evaluation Techniques

Following an ordinary program run, the user may decide to use one of the following special techniques, all of which are optional:

- Working with case distinctions.
- Automatic insertion of values.
- Editing of a formula.
- Transition to a *related graph*.

Working With Case Distinctions: was already explained and illustrated in Section 4.2 (see Step 6). The user is free to define a decomposition of the domain of a variable (up to nine segments). No closed interval is required—a decomposition can have forms such as $\omega = \{2, 3, \geq 4\}$ or $\{ 3 \leq \Delta \leq 6 , \Delta \geq 7 \}$. The decomposition into sub-classes, sub-sub-classes, …, is supported by the system up to four hierarchy levels; within the same hierarchy level up to nine descendants of the same direct ancestor are permitted. If the same improvement is achieved for all subcases of the same case, then this new knowledge can be *reached upward* to the next common ancestor. The idea behind this—supported by experience—is the chance that the same improvement can be derived in different ways within the different subcases. Optionally, the system can make proposals for plausible case distinctions.

Automatic Insertion of Values: was exemplified in Section 4.4. It should be supplemented here that no closed interval is required. If an integer variable is bounded only from one side, then the tentative insertion of numerical values starts at that bound, and continues for as long as the formula just considered leads to a contradiction and thus makes it possible to narrow that bound. For the case of very large intervals and/or real variables, heuristic procedures exist that supply preliminary data to the user, who has to decide whether a proposed problem reduction seems plausible.

Editing: is possible for each of the formulæ that were transformed by an inference run. The user can

- simplify an arithmetic expression by hand,
- insert numerical or Boolean values for a variable,
- delete an *if* part if it is considered true,
- replace a *then* part by *false*,
- delete a formula (e.g., if it is recognized that an *if* part cannot be satisfied, or that an inequality is inferior to another one—the latter point is supported by the system).

Transition to a Related Graph: Some successful proofs in graph theory show that a transition from the given class of graphs to another class—called *related graphs* in short—may be advantageous. Such a transition can be defined by any unique unary graph transformation. There are formulæ that connect variables of the original graphs with variables of the related graphs—an example is provided by the transition to a complementary graph using theorems of the Nordhaus–Gaddum type or formulæ like $\omega(G) = \beta_0(\overline{G})$. After an ordinary inference run that yields new information on the original class of graphs, the user may switch over to a class of related graphs in order to start an inference process with respect to that second class. Then the new knowledge about the second class of graphs can be automatically transferred back to the original class of graph. Transitions to complementary graphs and to line graphs are implemented in the system. The required *interconnection knowledge bases* exist; these are the two *auxiliary knowledge bases* mentioned previously. The user is free to define additional types of derived graphs; in this case, of course, a corresponding interconnection knowledge base must be set up.

5.3. Options for External Control of the Inference Process

The essential options for an external influence on the inference process are:

- Masking.
- Ranking the variables.
- Working with or without a derivation tree.
- Partitioning the knowledge base.

Masking: Each graph invariant can be *masked*; that is, it will be treated as inexistent during the same session. This tool is mainly used if the user is sure that a certain variable will not contribute to the solution. Also, each single statement can be masked; thus, for example, it is possible to make an inference run with or without use of the four-color theorem.

Ranking of the Variables: A ranking, that is, a linear order, of all graph invariants is defined. In the case where an equality between two numerical variables is derived in the inference process, this ranking determines whether x will be substituted for y or *vice versa*. The ranking also has an influence on the order within output lists. The user can alter the ranking individually and store the new ranking for future use.

Working With or Without a Derivation Tree: The user can decide whether or not a derivation tree is to be built up during an inference run. The derivation tree is required if the user later wants to obtain information about the way a certain result has been derived. Working without the derivation tree will reduce the program runtime.

Partitioning the Knowledge Base: In view of the extended knowledge bases, strategies can be recommended to speed up the inference process. The main knowledge base is partitioned in the following way. Each of its entries is assigned to one of three subsets whose members may be named *very important*, *important*, or *less important*. At the outset, only the very important class statements are used; in later inference runs those in the important class are included, until finally all entries in the knowledge base are active. In this way, some useful intermediate results can be achieved in earlier inference runs, with the consequence that many expressions can be simplified rather soon. Four different strategies for a partitioning of the knowledge base were empirically tested. It was found that in most cases this technique leads to a considerable reduction in computing time. The user may choose among these four strategies—in any case, one of them is predefined as a standard (according to the empirical results). For details consult the paper **[31]**.

5.4. Technical Details, References, Availability

Implementation of the system started in 1985 and continued until 2000. Since 2000, no further revision of the program was possible, but the knowledge bases are permanently updated. Due to side conditions about the year 1985 (students' knowledge and equipment, also administrative rules), the system was programmed in PASCAL (in the final stage about 30,000 lines of code) and on the basis of MS-DOS; the menus are in German. A transfer to modern computers is possible, and has already been successfully performed.

Additional information, for example, about examples of application, criteria for the selection of graph invariants and formulæ, parameter dependence of required computer runtime, and strategies to speed up computer runs with large knowledge bases, can be found in two papers **[32][33]** that contain many references. The executable program is freely available (email: **<t4141ax@mail.lrz-muenchen.de>**), as well as the published and unpublished expertise acquired in many years of practical work with the system.

6. Recent Little Steps and a Short Outlook

Small improvements that were found after finishing the main part of this paper are reported here without proof. Every counterexample to Reed's Conjecture must fulfil: $p \geq 14$, $q \geq 34$, $\alpha_0 \geq 10$, and $\Delta \leq p - 7$.

Future research on Reed's Conjecture may start from the abundant literature on color critical graphs, of which only a small proportion has been used to date. Another promising approach could be based on the complements of possible counterexamples—some structural properties of these complementary graphs are compiled here.

Practical use of the system *KBGRAPH* continues. The knowledge bases are permanently updated; but nevertheless the system should be reprogrammed totally from the beginning, free from restrictions imposed by earlier hardware, based upon a modern programming language and operating system, and exploiting the expertise accumulated over the years with respect to design, updating, and practical work.

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