## Proof and demonstration

- Consider proofs or demonstrations in mathematics

  **Claim.** The sum of any two even integers is even.

  *Proof.* Suppose $x$ and $y$ are even integers. Then, $x = 2a$ and $y = 2b$, for some integers $a$ and $b$ (by evenness). So $x + y = 2a + 2b$. By distributivity, $2a + 2b = 2(a + b)$, and so $x + y = 2a + 2b = 2(a + b)$. Therefore $x + y = 2(a + b)$ and by evenness it follows that $x + y$ is even. □

  Evenness: An integer $x$ is even if and only if $x = 2k$, for some integer $k$.

  Distributivity: $z(x + y) = zx + zy$

## Symbolic notation

- Given any two numbers, the result of adding them together in one order is the same as the result of adding them together in the reverse order.

- $x + y = y + x$

## A formal system of human knowledge

- In the western philosophical tradition there is a recurring idea that all concepts can be analysed as structures composed of the basic concepts (e.g. Aristotle, famously, held that man can be defined as "rational animal"), and therefore all truths can be encoded in a small set of concepts and composition rules.

- The dream: Somehow use the kind of precise notation from mathematics to develop a formal human language. And have a rigorous method of expressing and computing any truth.

  - e.g. Gottfried Leibniz's 17th century speculations about the development of an alphabet and algebra of human knowledge, with his *characteristica universalis* and *calculus ratiocinator*.
    - *Characteristica universals*: a universal formal language able to encode all human knowledge
    - *Calculus ratiocinator*: a formal inference engine able to compute all the truths from the basic truths.
  - e.g. Gottlob Frege's 19th century development of modern logic and the *Begriffsschrift*.

  - e.g. Bertrand Russell, the logical positivists, e.g. most ambitiously in Rudolf Carnap's *Der logische Aufbau der Welt*.

- We'd like to develop a symbolic language that reveals the logical form of sentences and arguments in terms of the shapes and arrangements of the symbols. And have a way to check for the validity or invalidity of an argument.

**Formal languages**

- A formal language is a set of strings of symbols (formulas or sentences)
- Can be completely defined without reference to an *interpretation*
  - symbols and formulas are abstract; tokens of symbols and formulas are marks on paper or chalkboards
- The set of formulas is stipulated by its creator by specifying:
  - *symbols*: the atomic expressions of the language
  - *formation rules*: say what strings of symbols are formulas (or well-formed)

**Language $W$**

- Symbols: $\triangle$, $\Box$,
- Formation rule: Any finite string of symbols of $W$ that begins with a '$\triangle$' is a formula

(Q1) Is '$\triangle\Box$' a formula?
(Q2) Is '$\triangle$' a formula?
(Q3) Is '$\triangle\triangle\triangle$' a formula?
(Q4) Is '$\triangle\Box\Diamond$' a formula?
(Q4) Is '$\triangle \rightarrow \Box$' a formula?

**Language $Y$**

- Symbols: $\triangle$, $\Box$, $\star$
- Formation rules:
  - $\triangle$ is a formula,
  - if $\phi$ is a formula, then $\phi \star \Box$ is a formula,
  - Nothing else is a formula.

(Q6) Is '$\star$' a formula of $Y$?
(Q7) Is '$\triangle \star \triangle$' a formula of $Y$?
(Q8) Is '$\square \star \triangle$' a formula of $Y$?
(Q9) Is $\triangle\square \star \triangle\square\triangle \star \square$ a formula of $Y$?
(Q10) Is $\triangle \star \square \star \square \star \square \star \square \star \square \star \square \star \square$ a formula of $Y$?
(Q11) Write a formula of $Y$ that is 5 symbols long: ____________________

**Formal languages**

- Basic symbols + Formation rules

**Formal systems**

- Formal language + Deductive apparatus
- (This generates a special subset of formulas that are the "theorems" of the system.)

**Deductive apparatus**

- Axioms, and/or Inference rules
  - *Axioms* are formulas that are "starting points", or "free theorems"
  - *Inference rules* specify what moves are allowed
- Notice that formal systems can have only axioms, only inference rules, or some of each

**Axiomatic systems versus Natural deduction systems**

- The early formalizations of sentential logic (and predicate logic) were axiomatic systems (e.g. Frege)
- These systems are ecomonical in that they need only specify a few axiom (schemas) and one inference rule, in order to generate all the desired theorems.
- But the derivations are cumbersome and don't correspond to our natural ways of thinking

**Natural deduction**

> "My starting point was this: The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. In contrast, I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a *calculus of natural deduction*..." [Gerhard Gentzen (1934) "Untersuchungen über das logische Schließen"]

In 1926 Łukasiewicz noted in his seminars that mathematicians do not actually construct their proofs by means of an axiomatic theory (in the style of Frege), but instead reason in more natural ways, e.g making assumptions and reasoning form there. His student Jaśkowski, developed the desired natural system in 'On the Rules of Suppositions in Formal Logic' (1934).

### System BANG

- Symbols: $\odot$, ■, !
- Formulation rules:
  - $\odot$ and ■ are formulas,
  - If $X$ is a formula, $!X$ is a formula.
- Axiom 1: $!!X$

### System Z

- Symbols: △, □
- Formulation rule: .
  - Any finite string (of Z) that begins with △ is a formula.
- Axiom 1: △□□□
- Rule 1:

$$\frac{\triangle\square\ldots}{\ldots\triangle\square}$$

(Q12) Is '△□△□' a theorem of $Z$?
(Q13) Is '□□△□' a theorem of $Z$?
(Q14) Is '△□□□' a theorem of $Z$?
(Q15) Derive: △□

### Derivation

- A *derivation* is an explicit line by line demonstration of how to produce a theorem according to to the rules and axioms of the formal system.
  - Note: Derivations are essentially "proofs" but there is a difference between *formal* proofs (derivations) and *informal* proofs. The "proofs" in maths are (usually) informal.