

Some useful concepts

Wolfgang Schwarz <wolfgang.schwarz@ed.ac.uk>

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University of Edinburgh

Preview

1. Sets
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3. Functions
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Sets

You want to talk about **several things** but use a singular expression?

Say: **the set of** so-and-sos.

- An object's essence is the set of properties it could not fail to have without ceasing to be that object.
- A moral code is a set of principles for living a just and good life.
- A theory is a set of sentences.

Notation:

$\{\text{Alice, Bob, Carol}\}$

$\{x : x \text{ is a city in France}\}$

$\{x \mid x \text{ is a city in France}\}$

The things considered together in a set are called **members** or **elements** of the set.

Notation: Paris \in {x : x is a city in France}

Sets do not have an order.

- $\{\text{Alice, Bob, Carol}\} = \{\text{Bob, Alice, Carol}\}.$

Sets can have sets as members.

If a theory is a set of sentences, then the set of all theories is a set of sets.

- Does this set have any sentences as members?

Let A and B be sets.

A is a **subset** of B ($A \subseteq B$) if all elements of A are elements of B .

The **union** $A \cup B$ of A and B is the set that contains all elements of A together with all elements of B .

The **intersection** $A \cap B$ of A and B is the set that contains all elements that are common to A and B .

It proves convenient to allow for sets with just one member, like {Paris}. Such sets are called **singletons**.

There is even an **empty set** \emptyset with no members at all.

- $\{x : x \text{ is a city in France} \} \cap \{x : x \text{ is a capital} \} = \{\text{Paris}\}$
- $\{x : x \text{ is a city in Switzerland} \} \cap \{x : x \text{ has more than 1 million inhabitants} \} = \emptyset$
- $\{x : x \text{ is a city with more than 100 million inhabitants} \} = \emptyset$

$\{ \text{Paris} \} \neq \text{Paris}$

$\emptyset \neq \{ \emptyset \}$

The cumulative hierarchy of pure sets

1. \emptyset

2. $\emptyset, \{\emptyset\}$

3. $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$

...

ω . $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \dots$

$\omega+1$. $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \dots$

.....

$\omega + \omega$. $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \dots$

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ω^ω . $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \dots$

.....

For almost^{almost} any things whatsoever, there is a set of just those things.

- { Julius Caesar, the moon, Beethoven's 5th symphony, the EU }

There is no set of all sets.

Lists

You want to talk about **several things in a fixed order**?

Say: **list** or **sequence** or **series** or *n*-tuple

A list with two elements is called an **(ordered) pair**.

A list with three elements is called a **triple**.

- Some pairs of hallucinations and perceptual experiences are indistinguishable for the subject.
- An utterance context is a triple of a possible world, an individual, and a time.

Functions

Functions

Given two sets A and B , a **function** (or **mapping**) from A (in)to B associates each element of A with a unique element of B .

Examples:

- the function (from the set of countries to the set of cities) that assigns to every country its capital.
- the function (from the set of people to the set of real numbers) that assigns to every person their height in meters.

If f is a function from A to B , then A is called the **domain** of f and B the **codomain**.

More examples:

- Propositions are functions from possible worlds to truth values.
- An interpretation is a function from words to meanings.
- $\sqrt{\cdot}$ is a **partial** function from real numbers to real numbers.

Lambda notation:

- $\lambda x.x^2$
- $\lambda x.\text{the capital of } x$

If the domain and codomain of a function are the same set, the function is called an **operation on** that set.

- Addition is an operation on the set of real numbers.
- Disjunction is an operation on the set of sentences.

Relations

A **(binary) relation** is something that can in some sense “obtain” between the elements of some set A and the elements of a possibly different set B .

- Paris stands in the capital relation to France.
- Edinburgh stands in the partner city relation to Krakow.
- France stands in the neighbour relation to Germany.
- David Trump stands in the denial relation to climate change.

For any set R of pairs whose first member is in A and whose second member is in B there is a relation from A to B , namely the relation of being paired by R .

Don't say stupid things:

Suppose that Lois is a monolingual German speaker [...] Lois bears no relation whatsoever to the English sentence 'Superman is strong'.

A set is **closed under** a relation R if whenever a is in the set and aRb , then b is in the set.

- Is knowledge closed under logical consequence?

A relation R is **transitive** if aRb and bRc entails aRc .

Are these transitive?

- loving
- being older than
- \in
- \subseteq

A relation R is **symmetric** if aRb entails bRa .

Are these symmetric?

- loving
- being older than
- being married to
- being less than 2km away from

A relation R is **reflexive** if it obtains between any thing (for which it is defined) and itself.

Are these reflexive?

- loving
- being the same age as
- being married to
- being less than 2km away from

Relations that are transitive, symmetric, and reflexive are called **equivalence relations**.

- Having (exactly) the same subject matter is an equivalence relation between propositions.

Relations that are transitive, asymmetric, and irreflexive are called **strict partial orders**.

- Being morally preferable is a strict partial order on possible acts.

A **relational structure** is a set of things that have certain properties and stand in certain relations to one another.

Two relational structures are **isomorphic** if they have the same structure; that is, if there is a function f from the one set onto the other such that whenever aRb then $f(a)Rf(b)$ and whenever $f(a)Rf(b)$.

Such a function is called an **isomorphism**.

Truths

Analytic truths are true in virtue of the meaning of the words they contain.

- All triangles have three sides.
- If Alice is Bob's mother, then Bob is Alice's child.

All other truths are **synthetic**.

- We're more than half way through this talk.
- I'm really looking forward to pizza.

A truth is **a priori** if it can in principle be known without relying on experience.

- All triangles have three sides.
- If Alice is Bob's mother, then Bob is Alice's child.

All other truths are **a posteriori**.

- We're more than half way through this talk.
- I'm really looking forward to pizza.

Are there analytic but a posteriori truths?

Are there synthetic but a priori truths?

- Nothing is both red and green all over.
- There is no largest prime number.

Possibilities

Philosophers love to talk about merely possible scenarios.

- The Trolley Problem
- Gettier cases
- The Experience Machine
- The Chinese Room
- Mary the colourblind neuroscientist
- ...

There are different **kinds of possibility**:

- **epistemic possibility**: compatible with our knowledge
- **physical possibility**: compatible with the laws of physics
- **logical possibility**: conceivable without contradiction
- **metaphysical possibility**: compatible with the laws of metaphysics or with the essence of things

Possibilities can be conjoined, disjoined, and negated.

Is possibility closed under conjunction?

No. Sometimes what you get by conjoining possibilities is an impossibility:

- The die lands on 2 and the die lands on 4.
- The die lands on 2 and it does not land on 2.

A possibility **entails** another if it is impossible for the first to be true and the second false.

- A runaway trolley is heading towards five people tied to the tracks. You can pull a lever to divert the trolley onto another track where it will run over one person.
- A runaway trolley is heading towards five philosophers tied to the tracks. You can pull a lever to divert the trolley onto another track where it will run over one physicist.

A possibility is **complete** if it entails A or $\neg A$ for every other possibility A .

Complete possibilities are also known as **possible worlds**.

Intuitively, a possible world is a fully specific way the universe might be – as detailed as the actual universe, but different in some way or other.

Two possibilities?

- A runaway trolley is heading towards five people tied to the tracks. You can pull a lever to divert the trolley onto another track where it will run over one person.
- Five people are tied to a railway track. A runaway trolley is heading towards them. You can pull a lever to divert the trolley onto another track. If the trolley goes on that track, it will run over one person.

A plausible identity condition

A is the same possibility as B if it is impossible for A to obtain without B , and for B to obtain without A .

Possibilities

The “plausible identity condition” supports the following assumptions:

1. Associativity: $A \vee (B \vee C) = (A \vee B) \vee C$
2. Commutativity: $A \wedge B = B \wedge A$
3. Absorption: $A \vee (A \wedge B) = A$
4. Identity: $A \vee (B \wedge \neg B) = A$
5. Idempotence: $A \vee A = A$
6. Distributivity: $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
7. Complementarity: $A \vee \neg A = B \vee \neg B$

These (together with their mirror image with all \wedge s and \vee s swapped) are the conditions on a **Boolean algebra**.

A lesson from universal algebra

The set of possibilities is isomorphic to the set of sets of possible worlds.

So we can think of possibilities as **sets of possible worlds**.

- Conjunction is union: $A \wedge B = A \cup B$
- Disjunction is intersection: $A \vee B = A \cap B$
- Negation is complementation: $\neg A = \{w : w \notin A\}$
- Entailment the subset relation: A entails $B \Leftrightarrow A \subseteq B$.

An application: **possible worlds semantics**

- 'It is possible that A ' is true iff A is true at some possible world.
- 'It is necessary that A ' is true iff A is true at all possible worlds.
- 'If A then B ' is true iff B is true at the closest world where A is true.

An application: **supervenience**

Property P **supervenes** on properties Q_1, Q_2, \dots if any possible worlds that are identical in the distribution of Q_1, Q_2, \dots are also identical in the distribution of P .

- Do mental properties supervene on physical properties?
- Do moral properties supervene on non-moral properties?

Probabilites

A **measure** is a function from a set of sets to the real numbers ≥ 0 such that whenever $A \cap B = \emptyset$ then the measure of $A \cup B$ is the sum of the measure of A and the measure of B .

A **probability measure** is a function P from the space of possibilities to the real numbers in the interval $[0, 1]$ such that

- $P(A \vee \neg A) = 1$
- if A entails $\neg B$ then $P(A \vee B) = P(A) + P(B)$.

Less formally:

- For any possibility A , it makes sense to ask about the probability of A .
- If A is necessary, it has probability 1.
- If A and B couldn't both be true, then the probability of either of them being true is the sum of their individual probabilities.

Probability has proved useful in many areas of philosophy.

- Bayesian epistemology is a branch of epistemology that takes seriously that rational beliefs come in degree; these degrees are modelled as probabilities.
- Bayesian models of cognition suggest that the brain is an essentially probabilistic engine.
- According to Bayesian confirmation theory, evidence E confirms a hypothesis H iff $P(H \wedge E) > P(H)P(E)$.

Often we don't know the precise value of some quantity, but we can assign probabilities to various possible values.

- A military intervention has an 80 percent chance of no civilian casualties and a 20 percent chance of 100 casualties.

The **expected value** (or **expectation**) of a quantity is the probability-weighted average of the possible values.

- The expected number of civilian casualties is $0 \times 0.8 + 100 \times 0.2 = 20$.

Another example:

- You consider not buying chicken from factory farms. Will fewer chickens be tortured and killed as a result?

Most likely not. Let's say:

- there's a 99.9 percent chance that your choice will not save any chicken;
- there's an 0.1 percent chance that your choice will lead to the downsizing of a chicken factory and save 10000 chicken.

Then the expected number of chickens you save is $0 \times 0.999 + 10000 \times 0.001 = 10$.

Phew!