Russell's Paradox

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Naïve set theory

Comprehension Axiom:

For every property, there is a set that contains all the objects that have this property.

- It is possible that some sets are not members of themselves.
- It is possible that some sets are members of themselves.

Russell's set

The set of all objects that do not contain themselves.

The property here is: *not containing itself*

So is R a member of itself?

- If we assume that it is, then R must not be a member of itself.
- From this: if R is not a member of itself, then R satisfies the condition to be a member of itself.
- Conclusion: R must simultaneously be a member and not be a member of itself - a contradiction.

Russell's Paradox

Premise 1: Russell's set is not a member of itself.

Premise 2: Russell's set is a member of itself.

Conclusion: Russell's set is both a member and not a member of itself.

The paradox relies on the intuitions of naive set theory.

Barber Example

- Town where:
  - Everyone must be clean shaven
  - There exists a barber shaves all and only those who do not shave themselves.

But then who shaves the Barber?

– Seems like he must both shave himself and not shave himself.

Possible Solution: There simply cannot be such a Barber.

This is unsatisfactory for Russell's Paradox

- Dismissing the existence of Russell's set is not as simple as dismissing the existence of such a barber.
- Russell's set still provides problems for the intuitions of naive set theory.

How to solve the paradox

- We want to say that R is not a possible set.
- 2 ways: the property *not containing itself* is not a well formed property OR the comprehension axiom is wrong.

Russell’s Theory of Types

- Russell’s theory of types aims to avoid the paradox by stipulating that no set can be a member of itself.
- This is because the set is not the right type of thing to be its own member.

Separation Axiom

The second way of saying that R is not a possible set: making the comprehension axiom stricter.

- Zermelo and Fraenkel:

  The only sets that exist are those sets that explicitly exist and further subsets of these sets.
- Require a pre-existing set *A* and some property to make a new subset.

SA and Russell's Set

- Assume that R is a member of itself: R must satisfy the condition of not being a member of itself. This is not possible, therefore R cannot be a member of itself.
- Assume that R is not a member of itself: R satisfies the second condition.

  Is R a member of A?

  Yes: Satisfies the first condition and means that R *is* in fact a member of itself.

  We have seen this is not possible in the first point, so R cannot be a member A.
- If R is not a member of A, then R cannot be a member of itself for this reason.
- Solved! R is not a member of itself *because it is not a member of the pre-existing set A*.

What do we think?

- Russell's theory of types
  - seems to solve the problem by drawing an unnatural distinction which we would not usually accept
- Where does set A come from? Set A is just any set that can explicitly exist - but cannot be a universal set or set of all sets.